ANALISI 2 | Gi 26/3/2015 |

Esercizi svolti

1. $\lim_{n\to\infty} \int_{\mathbb{R}} \frac{1}{\sqrt[n]{1+x^{2n}}}\,dx$.

L'integrando è pari, quindi il limite proposto diventa

$$\lim_{n\to\infty} 2\int_0^{\infty} \frac{1}{\sqrt[n]{1+x^{2n}}}\,dx.$$

Sia $f_n(x)=\frac{2}{\sqrt[n]{1+x^{2n}}}$. Si ha

$$\lim_{n\to\infty} f_n(x)=\begin{cases} 2 & \text{se } 0\le x\le 1 \\ 2/x^2 & \text{se } x>1 \end{cases},$$

inoltre

$$0\le f_n(x)\le \begin{cases} 2 & \text{se } 0\le x\le 1 \\ 2/x^2 & \text{se } x>1 \end{cases}$$

Poiché la funzione $g(x)=\begin{cases} 2 & \text{se } 0\le x\le 1 \\ 2/x^2 & \text{se } x>1 \end{cases}$ è sommabile in $[0,\infty[$, si conclude che

$$\lim_{n\to\infty} 2\int_0^{\infty} \frac{dx}{\sqrt[n]{1+x^{2n}}}\,dx = 2\int_0^1 dx + 2\int_1^{\infty} \frac{1}{x^2}\,dx = 4.$$

2. Per $a\in[-\infty,\infty[$ calcolare $\lim_{n\to\infty} \int_a^{\infty} \frac{xe^{-nx}}{1+e^{-nx}}\,dx$.

Sia $f_n(x)=\frac{xe^{-nx}}{1+e^{-nx}}$. Si ha, per $x\ge 0$, $f_n(x)\to 0$, mentre, per $x<0$, $f_n(x)=-|f_n(x)|=-\frac{|x|e^{n|x|}}{1+e^{n|x|}} \searrow -|x|=x$.

Per $x \geq 0$ si ha inoltre

$$0 \leq f_n(x) \leq x e^{-x} \quad (\text{sommabile in } [0,\infty[).$$

Dunque: se $a \geq 0$,

$$\int_a^\infty f_n(x)\,dx \longrightarrow \int_a^\infty 0\,dx = 0 \quad \text{per convergenza dominata;}$$

per $a < 0$,

$$\int_a^\infty f_n(x)\,dx = \int_a^0 f_n(x)\,dx + \int_0^\infty f_n(x)\,dx =$$

$$= -\int_a^0 |f_n(x)|\,dx + \int_0^\infty f_n(x)\,dx.$$

Per convergenza monotona,

$$\int_a^0 |f_n(x)|\,dx \to \int_a^0 |x|\,dx = \begin{cases} -\frac{a^2}{2} & \text{se } -\infty < a < 0 \\ -\infty & \text{se } a = -\infty \end{cases},$$

mentre naturalmente $\int_0^\infty f_n(x)\,dx \to 0$. In definitiva

$$\lim_{n\to\infty} \int_a^\infty \frac{x e^{-nx}}{1+e^{-nx}} = \begin{cases} 0 & \text{se } a \geq 0 \\ -\frac{a^2}{2} & \text{se } -\infty < a < 0, \\ -\infty & \text{se } a = -\infty. \end{cases}$$

3. $\displaystyle \lim_{n\to\infty} \int_0^2 \frac{nx + 3\sqrt{n}\,x^2}{(nx+2)\sqrt{|x-1|}}\,dx.$

Sia $f_n(x) = \dfrac{nx + 3\sqrt{n}\,x^2}{(nx+2)\sqrt{|x-1|}}$. Si ha

$$f_n(x) = \frac{n\left[x + 3x^2/\sqrt{n}\right]}{n\left(x + \frac{2}{n}\right)\sqrt{|x-1|}} \longrightarrow \frac{x}{x\sqrt{|x-1|}} = \frac{1}{\sqrt{|x-1|}} \quad \text{q.o. in } [0,2],$$

$$|f_n(x)| \leq \frac{1}{\sqrt{|x-1|}}\left[\frac{nx}{nx+2} + \frac{3\sqrt{n}\,x^2}{nx}\right] \leq \frac{1}{\sqrt{|x-1|}}\left[1 + \frac{3}{\sqrt{n}}x\right] \leq \frac{7}{\sqrt{|x-1|}},$$

e siccome $\frac{1}{\sqrt{|x-1|}}$ è sommabile in $[0,2]$, si conclude per convergenza dominata che

$$\lim_{n\to\infty}\int_0^2 \frac{nx+3\sqrt{n}\,x^2}{(nx+2)\sqrt{|x-1|}}\,dx=\int_0^2\frac{dx}{\sqrt{|x-1|}}=\int_{-1}^{1}\frac{dt}{\sqrt{|t|}}=$$

$$=2\int_0^1\frac{dt}{\sqrt{t}}=4.$$

4. Per $\alpha>1$ calcolare $\displaystyle\lim_{n\to\infty}\int_{1/n}^{n}\frac{|\sin x|^n}{x^\alpha}\,dx$.

Poniamo

$$f_n(x)=I_{[\frac{1}{n},n]}(x)\,\frac{|\sin x|^n}{x^\alpha};$$

si ha

$$f_n(x)\longrightarrow 0 \quad \text{q.o. in }]0,\infty[\quad (\text{in effetti, } \forall x\neq \tfrac{\pi}{2}+k\pi,\ k\in\mathbb{N})$$

e per ogni $n\geq\alpha$ vale

$$0\leq f_n(x)\leq \frac{|\sin x|^\alpha}{x^\alpha}\quad (\text{sommabile in }]0,\infty[\text{ essendo } \alpha>1).$$

Quindi, per convergenza dominata,

$$\lim_{n\to\infty}\int_{1/n}^{n}\frac{|\sin x|^n}{x^\alpha}\,dx=0.$$

Altri esempi:

- $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{nxe^x-e^{2x}}{x^2e^x+ne^{2x}}\,dx$
- (i) $\displaystyle\lim_{n\to\infty}\int_{\mathbb{R}}\left(\frac{\sin x}{x}\right)^n dx$, (ii) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sin x}{x}\right)^n=g(x)$ misurabile (iii) g non sommabile su $\mathbb{R}$.
- $\displaystyle\lim_{n\to\infty}\int_0^n\frac{\operatorname{arctg} x}{1+(n-x)^2}\,dx$
- $\displaystyle\lim_{n\to\infty}\int_0^\infty \sqrt[n]{n+x^2}\,e^{-\frac{nx}{n+1}}\,dx$.
- $\displaystyle\lim\int_0^1 f_n(x)dx$ con $\begin{cases} f_0(x)=x \end{cases}$ $x\in\,]0,1]$.