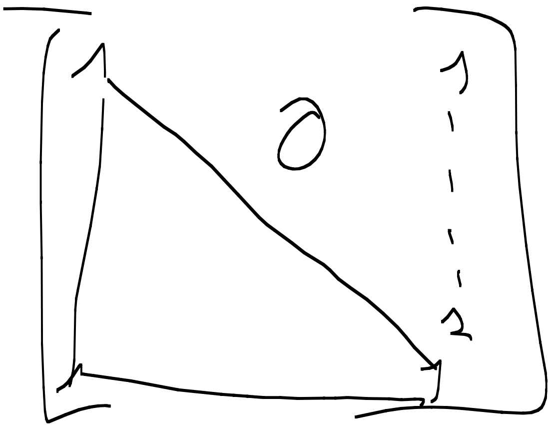
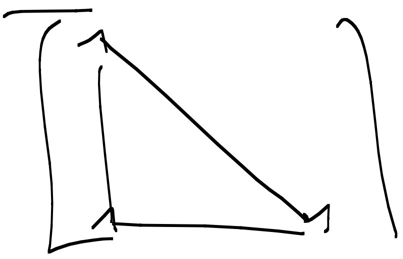
RICEVIMENTO 02/04

$$A = (a_{ij}) \in \mathbb{R}^{n \times n} \qquad a_{ij} = \begin{cases} 1 & \text{se } i \geq j \text{ o } j = n \\ 0 & \text{altrimenti} \end{cases}$$

$A =$ 

- Dire se A ammette fattorizzazione LU e in caso affermativo determinarla.

Sol: Le sottomatrici principali di testa di ordine $1 \dots n-1$ sono matrici triangolari inferiori con elementi diagonali $= 1$.

$A_k =$  $k = 1 \dots n-1$

$$A = \left(\begin{array}{ccc|c} 1 & & & 1 \\ \vdots & \ddots & & \vdots \\ 1 & \cdots & 1 & 1 \\ \hline 1 & \cdots & 1 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & & & \\ \vdots & \ddots & & 0 \\ 1 & \cdots & 1 & \\ \hline 1 & \cdots & 1 & 1 \end{array}\right) \left(\begin{array}{c|c} I_{n-1} & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & 0 \end{array}\right)$$

$$L_{n-1} \cdot U_{n-1} = \begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} I_{n-1}$$

$$x^T I_{n-1} = [1 \; \cdots \; 1] \iff x^T = [1 \; \cdots \; 1]$$

$$\begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} z = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \iff z = e_1$$

$$1 + \beta = 1 \iff \beta = 0$$

$\det A = 0$ / $\det A = 0$

$$\det A = \det U = 1 \cdot \ldots \cdot 1 \cdot 0 = 0$$

Sia $\|\cdot\|_V$ una norma vettoriale e $\|\ \|_M$ la norma matriciale indotta dalla norma vettoriale.

Allora $\forall x \in \mathbb{R}^n \ \forall A \in \mathbb{R}^{n \times n}$

$$\|A \cdot x\|_V \leq \|A\|_M \cdot \|x\|_V$$

Dim: $x = 0$

$$\|A \cdot x\|_V = 0 \qquad \|A\|_M \cdot \|x\|_V = \|A\|_M \cdot 0 = 0$$

ok

$x \neq 0$

$$\|Ax\|_V = \left\| A \underbrace{\frac{x}{\|x\|_V}}_{z} \|x\|_V \right\|_V$$

$$= \left\| A z \cdot \|x\|_V \right\|_V = \big| \|x\|_V \big| \, \|Az\|_V$$

$$= \|x\|_V \cdot \|Az\|_V \le \|x\|_V \cdot \|A\|_M$$

$$\|z\|_V = \left\| \frac{x}{\|x\|_V} \right\|_V = \frac{\|x\|_V}{\|x\|_V} = 1$$

Z versore

$$\|A\|_M = \max_{\|y\|_V = 1} \|Ay\|_V$$

$$\Rightarrow \|Az\|_V \le \|A\|_M$$

$$\begin{bmatrix} 2+\alpha & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2+\alpha \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Proviamo che per $\alpha > 0$ la matrice è invertibile

(1) Predominanza diagonale

$$|2+\alpha| > 1 \quad \text{e}$$
$$|2+\alpha| > 2$$

$\Rightarrow$ A è predominante diagonale $\Leftrightarrow$ $|2+\alpha| > 2$

$$\Leftrightarrow \begin{cases} 2+\alpha > 2 \\ 2+\alpha > 0 \end{cases} \cup \begin{cases} -2-\alpha > 2 \\ 2+\alpha \leq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha > 0 \\ \alpha \geq -2 \end{cases} \cup \begin{cases} -\alpha > 4 \Leftrightarrow \alpha < -4 \\ \alpha \leq -2 \end{cases}$$

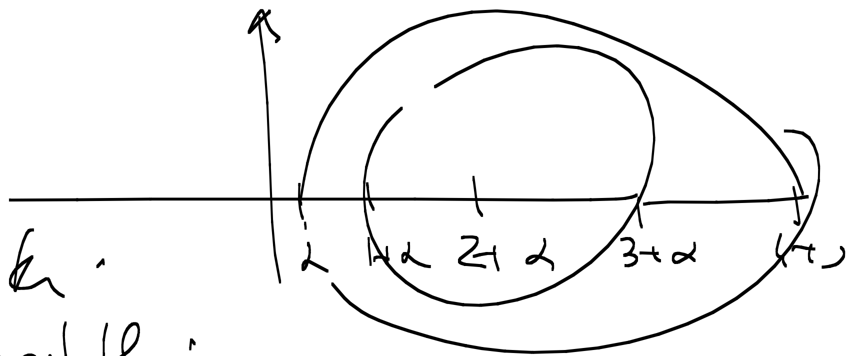
A p-diagonale $\boxed{\alpha > 0 \cup \alpha < -4}$

$\alpha \geq 0$ A p. diagonale e quindi invertibile

$$A = \begin{bmatrix} 2+\alpha & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2+\alpha \end{bmatrix}$$

$|2+\alpha| = 2+\alpha \geq 2$

$\alpha > 0$



$\alpha > 0$ $0 \notin \bigcup K_i$

$\Rightarrow$ A invertibile

$$A = \begin{bmatrix} -n & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & -n \end{bmatrix}$$

A invertibile ?

maggiorazioni per $k_2(A)$?

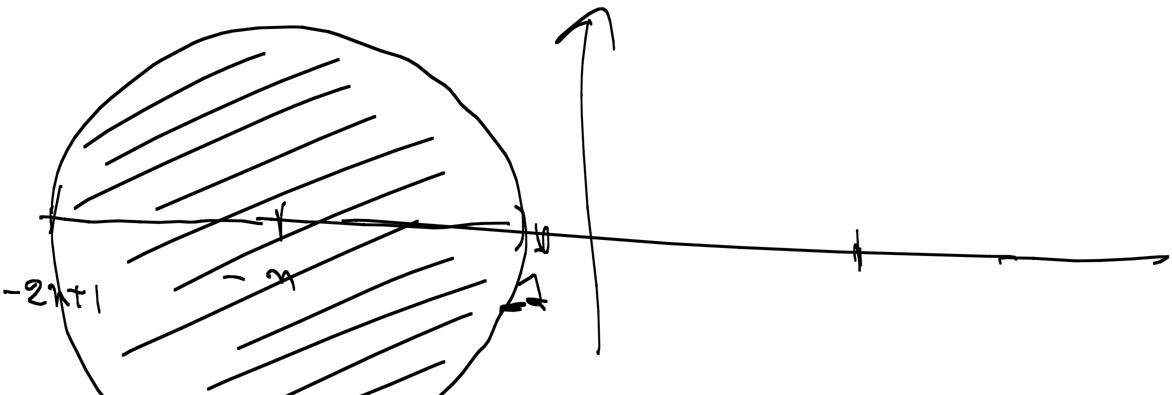
predominanza diagonale

$n = |-n| > n-1$ A predominante diagonale

quindi invertibile

$$\left(|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad i=1,\dots,n \right)$$

$K_2(A)$? Gershgorin.



$A = A^T \Rightarrow$ autovalori reali.

$-2n+1 \leq \lambda_i \leq -1$

$$\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{\rho(A^2)}$$

$$-2n+1 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_n \le -1$$

gli autoval di A^2 sono $\lambda_1^2, \lambda_2^2 \dots \lambda_n^2$

$$\Rightarrow 1 \le \lambda_n^2 \le \lambda_{n-1}^2 \le \dots \le \lambda_1^2 \le (1-2n)^2$$

$$\|A\|_2 = \sqrt{\rho(A^2)} = \sqrt{\lambda_1^2} = |\lambda_1|$$

$$\le 2n-1$$

$$\|A^{-1}\|_2 = \sqrt{\rho(A^{-T}A^{-1})} = \sqrt{\rho(A^{-2})} =$$

A^{-1} $\quad \frac{1}{\lambda_1} \dots \frac{1}{\lambda_n}$ $\quad = \sqrt{\left(\frac{1}{\lambda_n}\right)^2} = \left|\frac{1}{\lambda_n}\right|$

$A^{-2} = \left(\frac{1}{\lambda_1}\right)^2 \dots \left(\frac{1}{\lambda_n}\right)^2$ $\quad \le 1$

$$K_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = (2n-1) \cdot 1$$

$$= (2n-1)$$

(Matrice è ben condizionata)

$$A = \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}$$

(1) Passo della fattorizzazione

LU

$A = LU$

$$Ax = b \iff L\underbrace{Ux}_{y} = b$$

$$\iff \begin{cases} Ly = b \\ Ux = y \end{cases}$$

$$A = \begin{pmatrix} \theta & & 1 \\ & \ddots & \\ 1 & & \theta \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\iff \begin{cases} \theta x_1 + x_n = b_1 \\ \theta x_2 = b_2 \iff x_2 = b_2/\theta \\ \vdots \\ \theta x_{n-1} = b_{n-1} \iff x_{n-1} = b_{n-1}/\theta \\ x_1 + \theta x_n = b_n \end{cases}$$

$$\iff \begin{cases} x_i = b_i/\theta \quad i = 2 \dots n-1 \\ \theta x_1 + x_n = b_1 \quad x_n = b_1 - \theta x_1 \\ \theta x_n + x_1 = b_n \end{cases}$$

$$\theta(b_1 - \theta x_1) + x_1 = b_n$$

$$x_1(1-\theta^2) = b_n - \theta b_1$$

$$x_1 = (b_n - \theta b_1)/(1-\theta^2)$$

$$x_n = b_1 - \theta x_1$$

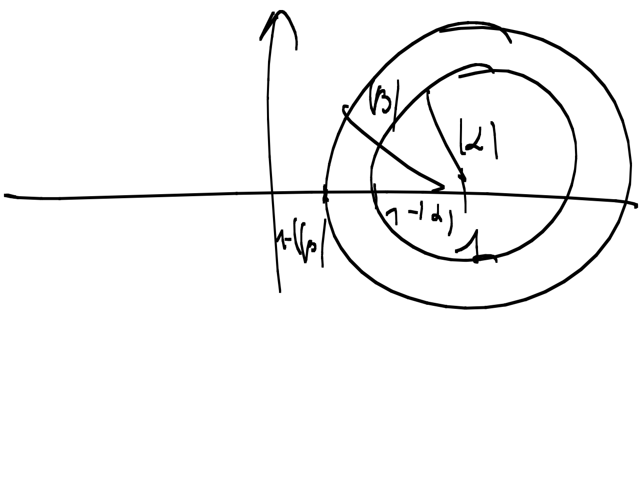
```
for k=2:n-1
    x(k)= b(k)/θ ;
end
x1= (b(n)-θ * b(1)) / (1-θ^2) ;
x(n)= b(1) - θ * x(1) ;
```

$O(n)$ operazioni aritmetiche

$$\begin{bmatrix}1 & & \beta\\ & \ddots & \\ \alpha & & 1\end{bmatrix}$$

determinare il numero δ t.c. A è invertibile

$\forall\ \alpha, \beta$ con $|\alpha| < \delta$ e $|\beta| < \delta$



Se $|\alpha| < 1$ e $|\beta| < 1$ allora A è invertibile

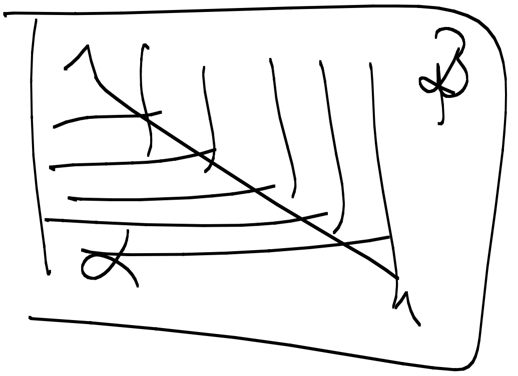
perché $0 \notin \bigcup K_i \Rightarrow A$ è invertibile

$\nexists$ possibile trovare $\delta > 1$ t.c. A è invertibile per ogni α, β

con $|\alpha| < \delta$ e $|\beta| < \delta$

No $\alpha = \beta = 1 \Rightarrow A$ è singolare

$\boxed{\delta = 1}$ la migliore stima



$A_k = I_k \quad k = 1 \dots n-1$

$\Rightarrow \forall \alpha, \beta \in \mathbb{R}$ A ammette fatt. LU

$$\begin{bmatrix} 1 & & \beta \\ & \ddots & \\ \alpha & & 1 \end{bmatrix} = \left(\begin{array}{c|c} I_{n-1} & \\ \hline \alpha \; 0 \cdots 0 & 1 \end{array}\right) \left(\begin{array}{c|c} I_{n-1} & \begin{matrix} \beta \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline & 1-\alpha\beta \end{array}\right)$$

$\alpha \cdot \beta + \gamma = 1 \qquad \gamma = 1 - \alpha\beta$

$\det A = \underline{1 - \alpha\beta}$ ((nessun pb !))

A è singolare $\iff 1 - \alpha\beta = 0$

$\iff \alpha \neq 0, \quad \beta \neq 0, \quad \alpha = \frac{1}{\beta}$

$$\begin{vmatrix} a & & 1 \\ & \ddots & \\ 1 & & a \end{vmatrix} \qquad \det A = a^{n-2}(a^2 - 1)$$

$$= a^n - a^{n-2}$$

<u>Controesempio</u>

$$|C_a| = \left| \frac{f'(a)}{f(a)} \cdot a \right|$$

$$= \left| \frac{n \cdot a^{n-1} - (n-2)\, a^{n-3}}{a^{n} - a^{n-2}} \right| \cdot |a|$$

$$= \left| \frac{n a^{2} - (n-2)}{a^{2} - 1} \right|$$

$|a| \to +\infty \qquad a^2 \to +\infty$

$$\lim_{|a| \to +\infty} |ea| = \lim_{|a| \to +\infty} \frac{\cancel{a^2} \cdot \left| n - \frac{n-2}{a^2} \right|}{\cancel{a^2} \cdot \left| 1 - \frac{1}{a^2} \right|}$$

$$= n$$

A simmetrica $\Rightarrow$ Teorema spettrale di diagonalizzabilità

$$\det(A - \lambda I) = \begin{pmatrix} a-\lambda & & 1 \\ & \ddots & \\ 1 & & a-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (a-\lambda)^{n-2}\left((a-\lambda)^2-1\right)$$

Gli zeri saranno $(a-\lambda)^{n-2} = 0 \iff \underline{\lambda = a}$

$(a-\lambda)^2 - 1 = 0 \iff (a-\lambda)^2 = 1$

$\iff a-\lambda = \pm 1 \qquad \lambda = \begin{matrix} a-1 \\ a+1 \end{matrix}$

gli autovalori di A sono
$\lambda = a \quad \sigma = n-2 \quad \tau = n-2$
$\lambda = a-1 \quad \sigma = 1 = \tau$
$\lambda = a+1 \quad \sigma = \tau = 1$

Sia $A \in K^{n\times n}$ $(\mathbb{C}^{n\times n})$. Definiamo

$$K_i := \left\{ z \in \mathbb{C} : |z - a_{ii}| \le \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \right\} \qquad i=1,\dots,n$$

Allora λ autovalore di $A \implies \lambda \in \bigcup_{i=1}^{n} K_i$

Rem: λ autovalor de $A \iff \exists x \neq 0$ tc $Ax = \lambda x$

$$Ax = \lambda x \iff \sum_{j=1}^{n} a_{ij} x_j = \lambda \cdot x_i \qquad i = 1 \ldots n$$

$$\iff \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j = (\lambda - a_{ii}) x_i \qquad i = 1 \ldots n$$

Posim $|x_p| = \|x\|_\infty = \max_{i=1..n} |x_i| \neq 0$

$$\Rightarrow \sum_{\substack{j=1 \\ j \neq p}}^{n} a_{pj} x_j = (\lambda - a_{pp}) x_p$$

$$\Rightarrow |(\lambda - a_{pp}) x_p| = \left| \sum_{\substack{j=1 \\ j \neq p}}^{n} a_{pj} x_j \right|$$

$$\Rightarrow |\lambda - a_{pp}| \cdot |x_p| \leq \sum_{j=1, j \neq p}^{n} |a_{pj}| |x_j| .$$

$$\Rightarrow |\lambda - a_{pp}| \le \sum_{\substack{j=1 \\ j \ne p}}^{n} |a_{pj}| \underbrace{\frac{|x_j|}{|x_p|}}_{\le 1}$$

$$\Rightarrow |\lambda - a_{pp}| \le \sum_{\substack{j=1 \\ j \ne p}}^{n} |a_{pj}|$$

$$\Rightarrow \lambda \in K_p \Rightarrow \lambda \in \bigcup_{i=1}^{n} K_i$$

$$f(x_1, \ldots, x_n) = 2(x_1 + x_2 + \cdots + x_n)$$

```
s = 0
for k = 1 : n
    s = s + x(k)
end
s = 2 * s
```

$O(n)$ operații

$O(1)$ op. ult

$$f(x_1, \dots, x_n) = \sum_{i=1}^{n} i \cdot x_i$$

```
s = 0
for k = 1 : n
    s = s + k * x(k)
end
```

$$A = \underline{\underline{\text{eye}(n)}} + \alpha \cdot \text{ones}(n,1) * \text{ones}(1,n)$$

$$= I + \alpha \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

$$= I + \alpha \begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha+1 & \alpha & \cdots & \alpha \\ \alpha & \ddots & & \vdots \\ \vdots & & \ddots & \alpha \\ \alpha & \cdots & \alpha & \alpha+1 \end{bmatrix}$$