Ricevimento 08/05

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ & \ddots & & \\ \frac{1}{2} & \cdots & \frac{1}{2} & 1 \end{bmatrix}$$

- <u>Convergenza Jacobi</u>: $n > 2$

$$J = \begin{bmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ & & & \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & \end{bmatrix} \quad \|J\|_1 = 1 \quad \|J\|_\infty > 1$$

$$= \frac{1}{2}(n-1)$$

$$\det(\lambda I - J) = \det \begin{pmatrix} \lambda & \frac{1}{2} & \cdots & \frac{1}{2} \\ \hline & \ddots & & \\ \frac{1}{2} & \cdots & \frac{1}{2} & \lambda \end{pmatrix}$$

$$= \lambda \cdot \det \begin{pmatrix} \lambda & \frac{1}{2} & \cdots & \frac{1}{2} \\ & \ddots & & \\ \frac{1}{2} & \cdots & \frac{1}{2} & \lambda \end{pmatrix} = \cdots = \lambda^{n-2} \det \begin{pmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{pmatrix}$$

$$\lambda^{n-2} \cdot \left(\lambda^2 - \frac{1}{4}\right) \begin{cases} \lambda = 0 \\ \lambda = \frac{1}{2} \\ \lambda = -\frac{1}{2} \end{cases}$$

$$\gamma_1(T) = \frac{1}{2} < 1 \Rightarrow \text{convergenza}$$

$$T^2 = \begin{bmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ & & & \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ & & & \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{4} & \text{—} & \frac{1}{4} & 0 \\ & & & \\ 0 & \frac{1}{4} & \text{—} & \frac{1}{4} \ \frac{1}{4} \end{bmatrix} \qquad \|T^2\|_1 = \frac{1}{2}$$

$$\frac{\|e^{(2k)}\|_1}{\|e^{(0)}\|_1} \le 2^{-k} \quad \forall k$$

$$e^{(m)} = T^m e^{(0)} \quad \forall m \ge 0$$

$$e^{(2k)} = T^{2k} \cdot e^{(0)}$$

$$T^{2k} = \underbrace{T^2 \cdot \ldots \cdot T^2}_{k \text{ volte}}$$

$$\|e^{(2k)}\|_1 = \|T^{2k} e^{(0)}\|_1 \leq \|T^{2k}\|_1 \|e^{(0)}\|_1$$

$$\leq \|T^2\|^k \|e^{(0)}\|_1$$

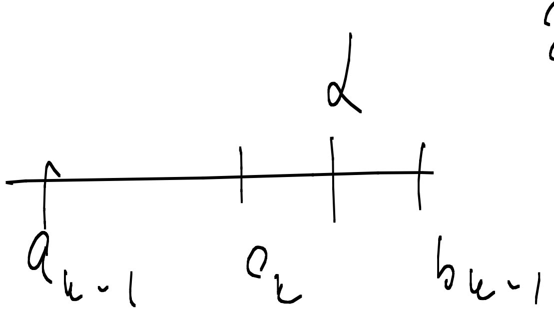
$$\frac{\|e^{(2k)}\|_1}{\|e^{(0)}\|_1} \leq \|T^2\|^k = \left(\frac{1}{2}\right)^k = 2^{-k}$$

$$f(x) = e^x - kx$$

$$[a_0, b_0] = [a, b]$$

$\{a_k\}_{k\geq 0}$ $\{b_k\}_{k\geq 0}$ $\{c_k\}_{k\geq 1}$

$$c_k = \frac{b_{k-1}+a_{k-1}}{2}$$



$$|\alpha - a_{k-1}| \leq \frac{b_{k-1}-a_{k-1}}{2^0} \leq \frac{b_{k-2}-a_{k-2}}{2^1} \dots \leq \frac{b_0-a_0}{2^{k-1}}$$

$$|\alpha - b_{k-1}| \leq \dots \frac{b_0-a_0}{2^{k-1}}$$

$$|\alpha - c_k| \leq \frac{b_0-a_0}{2^k}$$

$\hat{\alpha}$ tc $|\hat{\alpha} - \alpha| \leq \epsilon$

$$|\hat{\alpha}-\alpha| \leq \frac{b_0-a_0}{2^k} \leq \epsilon$$

$$\hat{\alpha} = c_k$$

$$\frac{b_0-a_0}{2^k} \leq \epsilon \iff 2^k \geq \frac{b_0-a_0}{\epsilon}$$

$$\iff k \geq \underbrace{\log_2\left(\frac{1}{\epsilon}\right)} + \log_2(b_0-a_0)$$

$$\boxed{\frac{b_0-a_0}{2^k} \geq \epsilon}$$

Attenzione!!

$$\boxed{\left(\frac{1}{2}\right)^k \geq \frac{\epsilon}{b_0-a_0}}$$

$$A = \begin{bmatrix} 1 & & & \alpha \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

α p.d. per i vari α

α Jacobi e Gauss-Seidel convergono.

$$G = \begin{bmatrix} 1 & & \\ -1 & \ddots & \\ & \ddots & \ddots \\ & -1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} & & -\alpha \\ & 0 & \\ & & \end{bmatrix}$$

$$= \left[\, 0 \,\middle|\, 0 \,\middle|\, \dots \,\middle|\, 0 \,\middle|\, v \,\right]$$

$$-\begin{bmatrix} 1 & & \\ -1 & \ddots & \\ & \ddots & \ddots \\ & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix} = v$$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} v = f \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$v_1 = \alpha \quad v_2 = \alpha \quad v_3 = \alpha \; \dots \quad v_n = \alpha$

$$G = \left[\begin{array}{c|c} 0 & \begin{matrix} -\alpha \\ \vdots \\ -\alpha \end{matrix} \end{array} \right]$$

G è triangolare superiore $\left\langle \begin{array}{l} \lambda = 0 \\ \lambda_n = -\alpha \end{array} \right.$

$\rho(G) = |-\alpha| = |\alpha| \qquad |\alpha| < 1$

$$F = \begin{bmatrix} 0 & & & -\alpha \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}$$

$$\det(\lambda I - C) = \det\begin{pmatrix} \lambda & & & \alpha \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & \lambda \end{pmatrix}$$

$$= \lambda \cdot \det\begin{pmatrix} \lambda & & \\ -1 & \ddots & \\ & -1 & \lambda \end{pmatrix} + (-1)^{n+1} \alpha \cdot \det\begin{pmatrix} -1 & \lambda & \\ & \ddots & \lambda \\ & & -1 \end{pmatrix}$$

$$= \lambda \lambda^{n-1} + (-1)^{n+1} \alpha (-1)^{n-1}$$

$$= \lambda^n + (-1)^{2n} \alpha$$

$$= \lambda^n + \alpha$$

$$\lambda^n + \alpha = 0 \iff \lambda^n = -\alpha$$

$$\Rightarrow \quad |\lambda|^n = |-\alpha| = |\alpha|$$

$$|\tilde{\lambda}| = \sqrt[n]{|\lambda|}$$

$$\varphi(T) = \sqrt[n]{|\lambda|} < 1 \Leftrightarrow |\lambda| < 1$$

$$\begin{bmatrix} n & -1 & \cdots & -1 \\ -1 & \ddots & & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \cdots & -1 & n \end{bmatrix} \qquad |n| = n > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| = n-1$$

$$O \perp$$

$$T = \begin{bmatrix} 0 & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \ddots & & \vdots \\ \vdots & & \ddots & \frac{1}{n} \\ \frac{1}{n} & \cdots & \frac{1}{n} & 0 \end{bmatrix} \qquad \|T\|_\infty = \frac{n-1}{n} < 1$$

$$\frac{\|e^{(k)}\|_\infty}{\|e^{(0)}\|_\infty} \leq 2^{-32}$$

$$\|e^{(k)}\|_\infty \le \|T^k\|_\infty \|e^{(0)}\|_\infty$$

$$\frac{\|e^{(k)}\|_\infty}{\|e^{(0)}\|_\infty} \le \|T^k\|_\infty \le \|T\|_\infty^k$$

$$\frac{\|e^{(k)}\|_\infty}{\|e^{(0)}\|_\infty} \le \left(\frac{n-1}{n}\right)^k$$

$$\left(\frac{n-1}{n}\right)^k \le 2^{-32}$$

$$k \log_2\left(\frac{n-1}{n}\right) \le -32$$

$$K \geq -\frac{32}{\log_2\left(\frac{n-1}{n}\right)}$$

$$\begin{bmatrix} a & & & \\ b & \ddots & & \\ c & \ddots & \ddots & \\ d & c & b & a \end{bmatrix}^{-1} = \begin{bmatrix} a' & & & \\ b' & \ddots & & \\ d' & \ddots & \ddots & \\ & c' & b' & a' \end{bmatrix}$$

Toeplitz

$$\begin{bmatrix} 1 & & \\ -1 & \ddots & \\ & \ddots & 1 \\ & & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ 1 & \ddots & \\ 1 & \cdots & 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha+1 & -1 & & -1 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ -1 & & -1 & \alpha+1 \end{bmatrix}$$

$$|\alpha+1| > 2$$

$$\begin{cases} \alpha+1 > 2 \\ \alpha+1 \geq 0 \end{cases} \cup \begin{cases} -\alpha-1 > 2 \\ \alpha+1 < 0 \end{cases}$$

$$\alpha > 1 \quad o \quad \alpha < -3$$

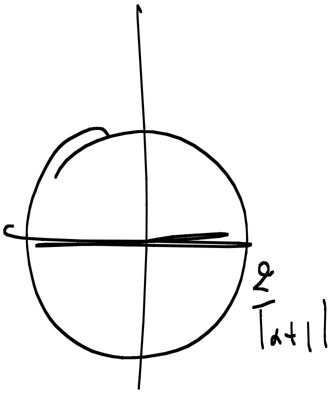
$$J = \begin{pmatrix} 0 & \frac{1}{\alpha+1} & & \frac{1}{\alpha+1} \\ \frac{1}{\alpha+1} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{\alpha+1} \\ \frac{1}{\alpha+1} & & \frac{1}{\alpha+1} & 0 \end{pmatrix}$$

$K_1 = K_2 = \dots = K_n$ sono cerchi di centro

e raggio $\dfrac{2}{|\alpha+1|}$

Condizione sufficiente è che

$$\frac{2}{|\alpha+1|} < 1 \iff |\alpha+1| > 2$$



$$J = \begin{bmatrix} 0 & \frac{1}{\alpha+1} & \cdots & \frac{1}{\alpha+1} \\ \frac{1}{\alpha+1} & \ddots & & \vdots \\ \vdots & & \ddots & \frac{1}{\alpha+1} \\ \frac{1}{\alpha+1} & \cdots & \frac{1}{\alpha+1} & \end{bmatrix}$$

$$J \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \frac{2}{\alpha+1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$\frac{2}{\alpha+1}$ è autovalore di J

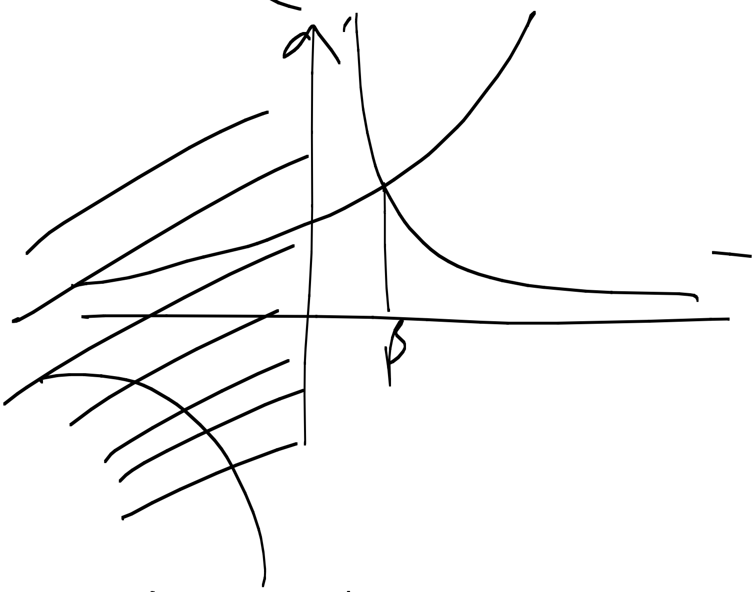
$\frac{2}{|\alpha+1|} < 1$ condizione necessaria e suff.

$$\rho(J) = \frac{2}{|\alpha+1|}$$

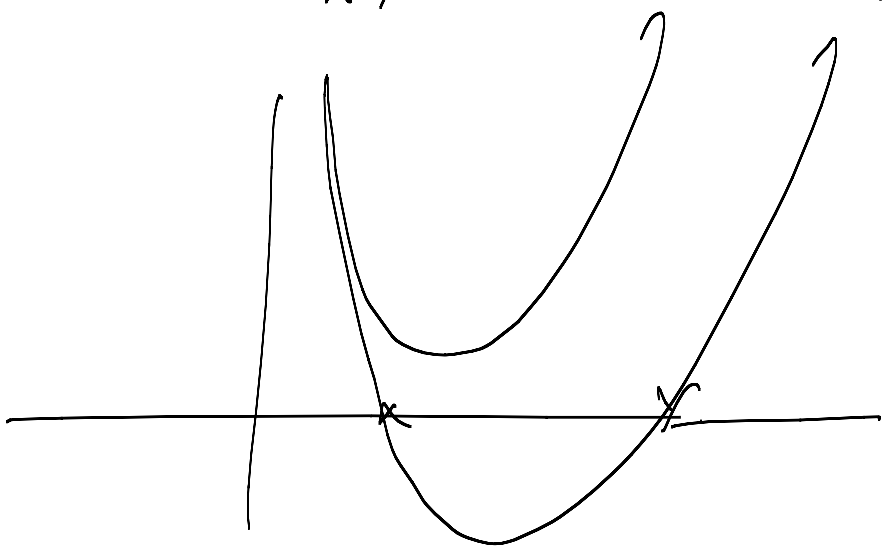
$$f(x) = e^x - \log x$$

$f'(x) = e^x - \frac{1}{x}$

$f'(x) \geq 0 \iff e^x \geq \frac{1}{x} \iff x \geq \beta$



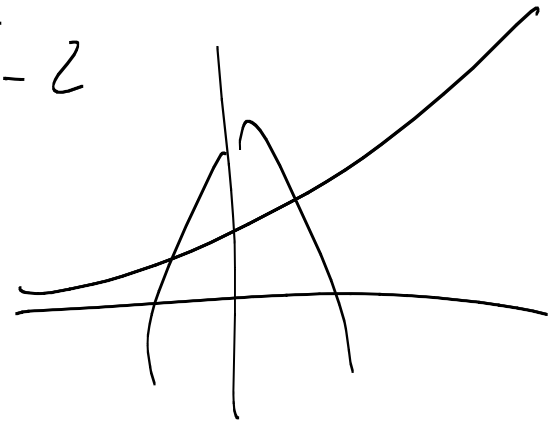
$f \in C^\infty(\mathbb{R}^+)$ $\lim_{x \to 0^+} f(x) = +\infty$ $\lim_{x \to +\infty} f(x) = +\infty$



$f(\beta_0) \geq 0$?

$f(x) = e^x + x^2 - 2$

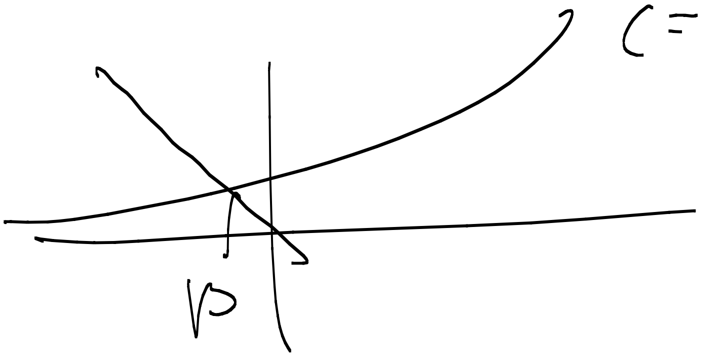
$e^x = 2 - x^2$



$f'(x) = e^x + 2x = 0$

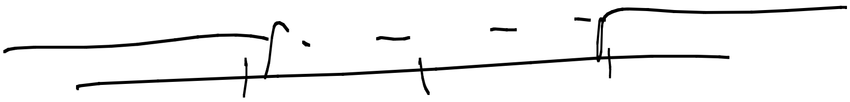
$e^x + 2x > 0 \iff e^x > -2x$

$\iff x > \beta$



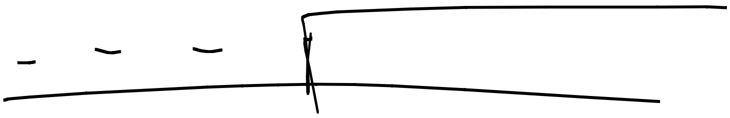
$e^\beta = -2\beta$

$f(\beta) = e^\beta + \beta^2 - 2 = \beta^2 - 2\beta - 2$



$f(x) = e^x + x^2 - 2 \qquad f \in C^\infty(\mathbb{R})$

$\lim_{x \to \pm\infty} f(x) = +\infty$



$f(\beta) < 0$