Linear Algebra: Exercises

Alessandro Berarducci, Oscar Papini

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Contents

Chapter 1

Linear Systems

Exercise 1.1. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}.$$

(a) Determine if the system $Ax = 0$ has zero, one or infinitely many solutions, and compute a basis of the space of solutions.

(b) Is it true that the system $Ax = b$ has a solution for any $b \in \mathbb{R}^3$? If so, prove the statement, otherwise find a counterexample.

Solution. (a) We have to find $\ker A$. At first we row-reduce A:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}$$

(second row) − (first row)
(third row) − 2(first row)

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(third row) − (second row)

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are two pivots and two free variables, therefore the system has infinitely many solutions. We choose x_2 and x_4 as free variables, and

- from $x_3 + x_4 = 0$ we get $x_3 = -x_4$;
- from $x_1 + x_2 + x_3 + x_4 = 0$ we get $x_1 = -x_2$.

Thus

$$\ker A = \left\{ \begin{bmatrix} -x_2 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} \middle| \, x_2, x_4 \in \mathbb{R} \right\}$$

and a basis is

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

(b) Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be a generic vector of $\mathbb{R}^3$. The system $Ax = b$ has a solution if and only if the matrix A and the complete matrix

$$\bar{A} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 1 & 1 & 2 & 2 & b_2 \\ 2 & 2 & 3 & 3 & b_3 \end{array}\right]$$

have the same rank. That happens if and only if the row reduced form of $\bar{A}$, which is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right],$$

has not a pivot in the third column, i.e. $b_3 - b_2 - b_1 = 0$. Any vector b *not* satisfying this condition, for example

$$b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

is a counterexample for the statement. ■

Exercise 1.2. Determine the number of solutions of the following system

$$\begin{cases} x + 2y - 3z = 4 \\ 4x + y + 2z = 6 \\ x + 2y + (a^2 - 19)z = a, \end{cases}$$

depending on the parameter $a \in \mathbb{R}$.

Solution. The matrix associated to the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 4 & 1 & 2 & 6 \\ 1 & 2 & a^2-19 & a \end{array}\right].$$

We get the row echelon form of the matrix subtracting the first row from the third, and then subtracting four times the first row from the second:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right].$$

Notice that the roots of $a^2 - 16 = (a + 4)(a - 4)$ are ± 4.

- If $a \neq \pm 4$, the term $a^2 - 16$ is not zero and we have a unique solution.
- If $a = -4$, the third equation of the system in echelon form becomes $0z = -8$ and there are no solutions.
- If $a = 4$, the third equation of the system in echelon form becomes $0z = 0$, which is satisfied by any value of z. Therefore the system has infinitely many solutions. ■

Exercise 1.3. Determine the number of solutions of the following system

$$\begin{cases} x_1 + kx_2 + (1+4k)x_3 = 1 + 4k \\ 2x_1 + (k+1)x_2 + (2+7k)x_3 = 1 + 7k \\ 3x_1 + (k+2)x_2 + (3+9k)x_3 = 1 + 9k \end{cases}$$

depending on the parameter $k \in \mathbb{R}$.

Solution. If we row-reduce the complete matrix of the system, we get

$$A = \left[\begin{array}{ccc|c} 1 & k & 4k+1 & 4k+1 \\ 0 & 1-k & -k & -k-1 \\ 0 & 0 & -k & -k \end{array}\right].$$

- If $k \neq 0$ and $k \neq 1$, the matrix A has three non-zero pivots in the first three columns. Therefore the coefficients matrix is invertible, and the system has a unique solution.
- If $k = 0$, the matrix A has two non-zero pivots in the first two columns, and we have only zeros in the third row. This means that the rank of the coefficients matrix is 2 and it is the same of the one of the complete matrix, but it is not the maximum rank. Thus the system has infinitely many solutions.

- If $k = 1$, the matrix A is no longer row reduced, so we can go on and get

$$\left[\begin{array}{ccc|c} 1 & 1 & 5 & 5 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

  This matrix has a pivot in the last column, which then cannot be a linear combination of the others. In this case the system has no solutions. ■

Exercise 1.4. Consider the following system, where the variables are x, y, z:

$$\begin{cases} (k+2)x + 2ky - z = 1 \\ x - 2y + kz = -k \\ y + z = k. \end{cases}$$

(a) For which values of $k \in \mathbb{R}$ does the system have a *unique* solution?

(b) For which values of $k \in \mathbb{R}$ does the system have *infinitely many* solutions? If any, compute for those values of k all the solutions of the system.

Solution. An $n \times n$ system has a solution if and only if the matrix of the coefficients and the complete matrix have the same rank; in particular, the solution is unique if this rank is n, and there are infinitely many solutions if this rank is less than n.

In this case, the complete matrix is

$$\left[\begin{array}{ccc|c} k+2 & 2k & -1 & 1 \\ 1 & -2 & k & -k \\ 0 & 1 & 1 & k \end{array}\right].$$

Elementary row operations change neither the set of the solutions, nor the ranks of the matrices. After some computation we get to the row echelon form:

$$\left[\begin{array}{ccc|c} k+2 & 2k & -1 & 1 \\ 1 & -2 & k & -k \\ 0 & 1 & 1 & k \end{array}\right]$$

permute rows

$$\left[\begin{array}{ccc|c} 1 & -2 & k & -k \\ 0 & 1 & 1 & k \\ k+2 & 2k & -1 & 1 \end{array}\right]$$

(third row) − (k + 2)(first row)

$$\left[\begin{array}{ccc|c} 1 & -2 & k & -k \\ 0 & 1 & 1 & k \\ 0 & 4k+4 & -(k+1)^2 & (k+1)^2 \end{array}\right]$$

(third row) − 4(k + 1)(second row)

$$\left[\begin{array}{ccc|c} 1 & -2 & k & -k \\ 0 & 1 & 1 & k \\ 0 & 0 & -(k+1)(k+5) & (k+1)(-3k+1) \end{array}\right]$$

- For $k \neq -1$ and $k \neq -5$, the coefficients matrix has maximum rank 3, so the system has a unique solution.
- For $k = -1$, both the rank of the coefficients matrix and the one of the complete matrix are equal to 2 and the system has infinitely many solutions.
- For $k = -5$, the coefficients matrix has rank 2 and the complete matrix has rank 3, so there are no solutions.

In order to conclude the exercise, we have to compute the set of the solutions when there are infinitely many ones, i.e. for $k = -1$. The reduced matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is associated to the system

$$\begin{cases} x - 2y - z = 1 \\ y + z = -1. \end{cases}$$

We can use z as a free variable and deduce the values of x and y in terms of it:

$$y = -1 - z, \quad x = 1 + 2(-1 - z) + z = -1 - z.$$

In conclusion, the set of solutions is

$$\{(-1-\lambda, -1-\lambda, \lambda) \mid \lambda \in \mathbb{R}\}. \qquad \blacksquare$$

Chapter 2

Vector Spaces

Exercise 2.1. **(I)** Find the coordinates of the vector $c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with respect to the basis (u_1, u_2) of $\mathbb{R}^2$, where

$$u_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(II) Find the coordinates of the polynomial $p(x) = 2 + x$ in the space $\mathbb{R}[x]_{\leq 1}$ of polynomials with real coefficients and degree less than or equal to 1, with respect to the basis $(q_1(x), q_2(x))$, where $q_1(x) = 3 + 2x$ and $q_2(x) = 2 + 3x$ (in this order).

Solution. **(I)** We have to find two numbers $x_1, x_2 \in \mathbb{R}$ such that $x_1 u_1 + x_2 u_2 = c$, i.e.

$$x_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

that is equivalent to solve the system

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We row-reduce the complete matrix

$$\left[\begin{array}{cc|c} 3 & 2 & 2 \\ 2 & 3 & 1 \end{array}\right] \quad \rightsquigarrow \quad \left[\begin{array}{cc|c} 3 & 2 & 2 \\ 0 & 5/3 & -1/3 \end{array}\right].$$

From the second equation we obtain $x_2 = -1/5$, which we substitute in the first equation to get, after a quick computation, $x_1 = 4/5$. Therefore the coordinates of c are $(4/5, -1/5)$.
(II) We use the basis $(e_1(x), e_2(x))$ of $\mathbb{R}[x]_{\leq 1}$, where $e_1(x) = 1$ and $e_2(x) = x$, to transfer the problem in $\mathbb{R}^2$:

- $p(x) = 2 + x = 2e_1(x) + e_2(x)$ which corresponds to the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$;

- $q_1(x) = 3 + 2x = 3e_1(x) + 2e_2(x)$ which corresponds to the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$;
- $q_2(x) = 2 + 3x = 2e_1(x) + 3e_2(x)$ which corresponds to the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

The system to solve is

$$x_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

which is the same of part **(I)**: once again, the solution is $(4/5, -1/5)$. *Note that this <u>does not</u> correspond to the polynomial* $(4/5)+(-1/5)x$: the coordinates are a pair of numbers, not a polynomial, and they just mean that $p(x) = (4/5)q_1(x) + (-1/5)q_2(x)$. ■

Exercise 2.2. Prove the following statement, or find a counterexample: if V is a $\mathbb{K}$-vector space and $v_1, v_2, v_3 \in V$ are vectors such that

(i) $v_1 \neq 0$;

(ii) $v_2 \notin \operatorname{span}(v_1)$;

(iii) $v_3 \notin \operatorname{span}(v_1, v_2)$;

then v_1, v_2 and v_3 are linearly independent.

Solution. The statement is true, and we will use *reductio ad absurdum* to prove it. Suppose that v_1, v_2 and v_3 are linearly dependent, that is there exist $x_1, x_2, x_3 \in \mathbb{K}$ such that

$$x_1v_1 + x_2v_2 + x_3v_3 = 0 \qquad (*)$$

and at least one of them is different from zero.

- If $x_3 \neq 0$, from $(*)$ we obtain

  $$v_3 = -\frac{x_1}{x_3}v_1 - \frac{x_2}{x_3}v_2$$

  which contradicts (iii).
- If $x_3 = 0$ but $x_2 \neq 0$, $(*)$ becomes $x_1v_1 + x_2v_2 = 0$ and, since $x_2 \neq 0$, we obtain

  $$v_2 = -\frac{x_1}{x_2}v_1$$

  which contradicts (ii).
- Finally, if $x_3 = 0$ and $x_2 = 0$, necessarily $x_1 \neq 0$; now $(*)$ has the form $x_1v_1 = 0$ from which we deduce $v_1 = 0$, contradicting (i).

Since all cases brought us to an absurd, the statement is proven. ■

Exercise 2.3. Let

$$v_1 = \begin{bmatrix}1\\1\\1\\2\end{bmatrix}, \quad v_2 = \begin{bmatrix}1\\2\\-1\\1\end{bmatrix}, \quad v_3 = \begin{bmatrix}0\\1\\1\\2\end{bmatrix}.$$

(a) Is it true that v_1, v_2 and v_3 are linearly independent, seen as vectors in $\mathbb{R}^4$? Write a basis of $\operatorname{span}(v_1, v_2, v_3)$ and complete it to a basis of $\mathbb{R}^4$.

(b) Is it true that v_1, v_2 and v_3 are linearly independent, seen as vectors in $(\mathbb{Z}_3)^4$? Write a basis of $\operatorname{span}(v_1, v_2, v_3)$ and complete it to a basis of $(\mathbb{Z}_3)^4$.

Solution. (a) In order to test the linear dependence, we put the three vectors in a matrix and row-reduce it:

$$\begin{bmatrix}1&1&0\\1&2&1\\1&-1&1\\2&1&2\end{bmatrix}$$

(second row) − (first row)
(third row) − (first row)
(fourth row) − 2(first row)

$$\downarrow$$

$$\begin{bmatrix}1&1&0\\0&1&1\\0&-2&1\\0&-1&2\end{bmatrix}$$

(third row) + 2(second row)
(fourth row) + (second row)

$$\downarrow$$

$$\begin{bmatrix}1&1&0\\0&1&1\\0&0&3\\0&0&3\end{bmatrix} \qquad (\star)$$

(fourth row) − (third row)

$$\downarrow$$

$$\begin{bmatrix}1&1&0\\0&1&1\\0&0&3\\0&0&0\end{bmatrix}$$

Since in the row reduced form there are three pivots, v_1, v_2 and v_3 are linearly independent over $\mathbb{R}$ and they are a basis of their span.

Now, to complete them to a basis of $\mathbb{R}^4$, we add a system of generators

$$A = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

and row-reduce the matrix, obtaining

$$\left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{array}\right].$$

The pivots are in the first, second, third and fifth columns, therefore the corresponding columns of A, i.e.

$$\left(v_1, v_2, v_3, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right),$$

form a basis of $\mathbb{R}^4$.

(b) Over $\mathbb{Z}_3$, the previous row reduction is still valid until we reach the matrix $(\star)$. From there, since $3 = 0$ in $\mathbb{Z}_3$, we can write $(\star)$ as

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and that is the row reduced form. In this case, we have only two pivots, so the vectors are linearly dependent and a basis of $\mathrm{span}(v_1, v_2, v_3)$ is (v_1, v_2), i.e. the vectors corresponding to the pivots.

As before, to complete it to a basis of $(\mathbb{Z}_3)^4$ we add a set of generators and row-reduce the resulting matrix (remember that all operations are done in $\mathbb{Z}_3$):

$$B = \left[\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

The pivots are in the first, second, fourth and fifth columns, therefore the corresponding columns of B, i.e.

$$\left(v_1, v_2, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right),$$

form a basis of $(\mathbb{Z}_3)^4$. ■

Chapter 3

Linear Subspaces

Exercise 3.1. Let $\mathbb{K}$ be a field, and let $V = \mathbb{K}[x]_{\leq 3}$. Consider the subspace $W \subseteq V$ given by

$$W = \{p \in V \mid p(1) = 0\}.$$

(a) Compute $\dim W$, justifying the answer.

(b) Compute the cardinality of W for $\mathbb{K} = \mathbb{Z}_{13}$.

Solution. (a) We know that $\dim V = 4$. It is clear that W is a subspace of V:

- the polynomial 0 belongs to W;
- if $p, q \in W$, then $(p+q)(1) = p(1) + q(1) = 0 + 0 = 0$, so $p + q \in W$;
- if $p \in W$ and $\lambda \in \mathbb{K}$, then $(\lambda p)(1) = \lambda \cdot p(1) = \lambda \cdot 0 = 0$, so $\lambda p \in W$.

Now, if $p \in V$, we can write $p = ax^3 + bx^2 + cx + d$ with $a, b, c, d \in \mathbb{K}$, and

$$W = \{ax^3 + bx^2 + cx + d \in V \mid a + b + c + d = 0\}.$$

If we identify V with $\mathbb{K}^4$ through the isomorphism that sends a polynomial to the vector of its coefficients, we can read W as a subspace of $\mathbb{K}^4$:

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{K}^4 \;\middle|\; a + b + c + d = 0 \right\}.$$

It is easy to see that we can use b, c and d as free variables and set $a = -b-c-d$; in other words

$$W = \left\{ \begin{bmatrix} -b-c-d \\ b \\ c \\ d \end{bmatrix} \;\middle|\; b, c, d \in \mathbb{K} \right\},$$

thus $\dim W = 3$.
(b) Every n-dimensional $\mathbb{K}$-vector space is isomorphic to $\mathbb{K}^n$, therefore if $\mathbb{K} = \mathbb{Z}_{13}$ we have $W \simeq (\mathbb{Z}_{13})^3$ and its cardinality is 13^3. ■

Exercise 3.2. Let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

be vectors in $\mathbb{R}^3$.

(a) Let $V = \operatorname{span}(v_1, v_2)$ and $W = \operatorname{span}(w_1, w_2)$. Find a basis of $V \cap W$.

(b) Complete $\{w_1, w_2\}$ to a basis of $\mathbb{R}^3$.

(c) Is it possible to complete $\{v_1, v_2\}$ to a basis of $\mathbb{R}^3$?

Exercise 3.3. Let $V = \operatorname{span}(v_1, v_2, v_3)$ (over $\mathbb{R}$), where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}.$$

Let W be the *subset* of V which contains all and only the vectors of V that have the first two components equal to 0.

(a) Compute $\dim V$.

(b) Is W a subspace of V?

(c) Compute a basis of W.

Solution. (a) We have to extract a subset of $\{v_1, v_2, v_3\}$ made of linearly independent vectors. We put the three vectors in a matrix and compute its row reduced form:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there are three pivots, the vectors v_1, v_2 and v_3 are linearly independent and form a basis of V, whose dimension is 3.

(b) The set W is

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in V \;\middle|\; x_1 = x_2 = 0 \right\}$$

and it is indeed a linear subspace of W.

- $0 \in W$ because $0 \in V$ (since V itself is a vector space) and the first two components of 0 are zeros (actually all components...).

- Let $w_1, w_2 \in W$, that is $w_1, w_2 \in V$ and
$$w_1 = \begin{bmatrix} 0 \\ 0 \\ a \\ b \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix}$$
for some $a, b, c, d \in \mathbb{R}$. Then $w_1 + w_2 \in V$ because V is a vector space, and
$$w_1 + w_2 = \begin{bmatrix} 0 \\ 0 \\ a+c \\ b+d \end{bmatrix}$$
so it belongs to W.
- Let $w \in W$ and $k \in \mathbb{R}$. With the same argument as above, $kv \in V$ (because $v \in V$) and its first two components are $k \cdot 0 = 0$, so $kv \in W$.

(c) Notice that $W = V \cap Z$, where Z is the set of the solutions of the system
$$\begin{cases} x_1 = 0 \\ x_2 = 0. \end{cases}$$
Then we look for a system of equations for V: a generic vector belongs to V if and only if
$$\operatorname{rk} \left[\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ 1 & 2 & 1 & x_2 \\ 1 & 2 & 2 & x_3 \\ 1 & 3 & -1 & x_4 \end{array}\right] < 4.$$
We row-reduce the matrix, obtaining
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ 0 & 1 & 2 & x_4 - x_1 \\ 0 & 0 & 1 & x_3 - x_1 \\ 0 & 0 & 0 & x_2 - x_1 \end{array}\right],$$
thus its rank is not 4 if and only if $x_2 - x_1 = 0$. Therefore $W = V \cap Z$ is the set of the solutions of
$$\begin{cases} x_2 - x_1 = 0 \\ x_1 = 0 \\ x_2 = 0. \end{cases}$$
Now, by substitution we have that the first equation is redundant, because it reduces to $0 = 0$. So actually $W = V \cap Z = Z$ and a basis of it is
$$\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$
■

Exercise 3.4. In the vector space $\mathbb{R}^4$, consider the subspace V given by the solutions of the system

$$\begin{cases} x + 2y + z = 0 \\ -x - y + 3t = 0 \end{cases} \tag{$*$}$$

and the subspace W generated by the vectors

$$w_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w_2 = \begin{bmatrix} 3 \\ -2 \\ -2 \\ 0 \end{bmatrix}.$$

Compute $\dim(V \cap W)$ and $\dim(V + W)$.

Solution. The two vectors spanning W are linearly independent, so $\dim W = 2$. On the other hand, it is easy to show that $\dim V = 2$, because we can choose x and y freely, and then z and t are uniquely determined. Moreover $W \not\subseteq V$ because, for instance, $w_1 \notin V$ ($x = 2$, $y = 0$, $z = 1$ and $t = 1$ is not a solution of the system ($*$)). This means that $\dim(V \cap W) < \dim W$ strictly, that is to say that either $\dim(V \cap W) = 1$ or 0. Notice that also $w_2 \notin V$, but this tells us nothing more about $\dim(V \cap W)$. Then, we have to check when a linear combination of w_1 and w_2,

$$a \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ -2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a + 3b \\ -2b \\ a - 2b \\ a \end{bmatrix}, \tag{$\dagger$}$$

belongs to V. By substituting the respective values in the system ($*$), we obtain

$$\begin{cases} (2a + 3b) + 2(-2b) + (a - 2b) = 0 \\ -(2a + 3b) - (-2b) + 3a = 0 \end{cases}$$

which can be simplified to

$$\begin{cases} 3a - 3b = 0 \\ a - b = 0 \end{cases}$$

that is equivalent to $a = b$. In other words, any vector of the form ($\dagger$) with $a = b$ belongs to $V \cap W$; in particular, if $a = b = 1$ we get the vector

$$\begin{bmatrix} 5 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

which is not the zero vector and belongs to $V \cap W$, thus proving that $\dim(V \cap W) \geq 1$. Since we already know that $\dim(V \cap W) \leq 1$, we can conclude that $\dim(V \cap W) = 1$.

Now the dimension of $V+W$ can be computed easily by Grassmann's Formula:

$$\dim(V+W) = \dim V + \dim W - \dim(V \cap W) = 2 + 2 - 1 = 3. \qquad \blacksquare$$

Exercise 3.5. Let $V \subseteq \mathbb{R}^4$ be the subspace $V = \operatorname{span}(v_1, v_2)$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix},$$

and let $W \subseteq \mathbb{R}^4$ be the subspace given by the solutions of the system

$$\begin{cases} x_1 + x_2 + 2x_4 = 0 \\ 2x_1 + x_2 - x_3 = 0. \end{cases} \tag{$*$}$$

Find a basis of $V \cap W$ and a basis of $V + W$.

Solution. In order to find $V \cap W$, we look for a system of equations for V: a generic vector of $\mathbb{R}^4$ belongs to V if and only if the two matrices

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 1 & 3 & x_2 \\ 1 & 1 & x_3 \\ 0 & 0 & x_4 \end{array}\right]$$

have the same rank. We row-reduce the second one, obtaining

$$\left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & 2 & x_2 - x_1 \\ 0 & 0 & x_3 - x_1 \\ 0 & 0 & x_4 \end{array}\right],$$

and this matrix must not have a pivot in the third column; therefore a set of equations for V is

$$\begin{cases} x_3 - x_1 = 0 \\ x_4 = 0. \end{cases} \tag{$\dagger$}$$

Putting together the systems ($\dagger$) and ($*$) gives a system of equations for $V \cap W$:

$$\begin{cases} x_3 - x_1 = 0 \\ x_4 = 0 \\ x_1 + x_2 + 2x_4 = 0 \\ 2x_1 + x_2 - x_3 = 0 \end{cases}$$

with associated matrix

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & -1 & 0 \end{bmatrix}.$$

The row reduced form of A is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rightsquigarrow \quad \begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ x_4 = 0. \end{cases}$$

We keep x_3 as a free variable, thus we have $x_1 = x_3$ and $x_2 = -x_3$, i.e.

$$V \cap W = \left\{ \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} \,\middle|\, x_3 \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

and that is a basis of $V \cap W$.

Now we turn to $V + W$. First of all, we compute a set of generators for W solving the system $(*)$: if we proceed as above, we get

$$W = \left\{ \begin{bmatrix} x_3 + 2x_4 \\ -x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} \,\middle|\, x_3, x_4 \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right).$$

Therefore

$$V + W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right)$$

and we have to extract a set of linearly independent vectors. To do so, we put the generators in a matrix and we row-reduce it:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 3 & -1 & -4 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & -2 & -6 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivots are in the first, second and fourth columns, thus a basis of $V + W$ is

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right).$$

■

Exercise 3.6. Consider the following subspaces of $\mathbb{R}^4$:

$$U = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right) \quad \text{and} \quad W = \operatorname{span}\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\\2\end{bmatrix}\right).$$

(a) Find a basis of $U + W$ and a basis of $U \cap W$.

(b) Does it exist a subspace $Z \subseteq \mathbb{R}^4$ such that $U \oplus Z = W \oplus Z = \mathbb{R}^4$? If so, determine this subspace, otherwise prove that it can't exist.

Solution. (a) We know that, if $\mathcal{G}_U$ is a set of generators for U and $\mathcal{G}_W$ is a set of generators for W, then $\mathcal{G}_U \cup \mathcal{G}_W$ is a set of generators for $U + W$, from which we can extract a basis. Therefore we put all the generators in a matrix

$$\begin{bmatrix}1 & 1 & 1 & 1\\1 & -1 & 0 & 2\\1 & 1 & 1 & 0\\1 & -1 & 0 & 2\end{bmatrix}$$

and reduce it in row echelon form. The result is

$$\begin{bmatrix}1 & 1 & 1 & 0\\0 & 2 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix};$$

the pivots are in the first, second and fourth columns, so the vectors

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \quad \begin{bmatrix}1\\2\\0\\2\end{bmatrix}$$

are linearly independent and form a basis of $U + W$.

Now notice that the generators of U are linearly independent, and so are the generators of W, therefore $\dim U = \dim W = 2$. From Grassmann Formula it follows that

$$\dim(U \cap W) = \dim U + \dim W - \dim(U + W) = 1.$$

thus any non-zero vector of $U \cap W$ forms a basis of it. For example,

$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \in U \cap W,$$

because it is one of the given generators of W and

$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right) \in U.$$

(b) If we complete a basis of U to a basis of $\mathbb{R}^4$, the added vectors span a complement of U, and the same is true for W. We are done if we can find two vectors w_1 and w_2 such that both

$$\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, w_1, w_2\right) \quad \text{and} \quad \left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\\2\end{bmatrix}, w_1, w_2\right)$$

are bases of $\mathbb{R}^4$; the requested subspace Z exists if and only if it is possible to choose such vectors, and in that case $Z = \operatorname{span}(w_1, w_2)$. It is easy to verify that the vectors

$$w_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \text{and} \quad w_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

satisfy the condition above. ■

Chapter 4

Linear Maps

Exercise 4.1. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map such that

$$f\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix} \quad \text{and} \quad f\left(\begin{bmatrix}2\\3\\4\end{bmatrix}\right) = \begin{bmatrix}2\\4\end{bmatrix}.$$

(a) Compute $f\left(\begin{bmatrix}8\\13\\18\end{bmatrix}\right)$.

(b) Compute the dimension of $\ker f$.

Exercise 4.2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map defined as

$$f\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x+2y\\2x+4y\\x+ay\end{bmatrix}$$

where $a \in \mathbb{R}$ is a parameter.

(a) Find the matrix $[f]$ associated to f with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

(b) For which values of a is f injective?

(c) For which values of a is f surjective?

Solution. (a) If (e_1, e_2) is the standard basis of $\mathbb{R}^2$, the i-th column of $[f]$ is the vector of the coordinates of $f(e_i)$ with respect to the standard basis of $\mathbb{R}^3$. Since

$$f(e_1) = f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\1\end{bmatrix} \quad \text{and} \quad f(e_2) = f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\4\\a\end{bmatrix},$$

we have

$$[f] = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & a \end{bmatrix}.$$

(b) The map f is injective if and only if $\ker f = \{0\}$. To compute $\ker f$, we row-reduce $[f]$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & a \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 2 \\ 0 & a-2 \\ 0 & 0 \end{bmatrix}.$$

The corresponding system has a unique solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if $a \neq 2$, and infinitely many solutions otherwise. So f is injective if and only if $a \neq 2$.
(c) Recall that $\operatorname{Im} f = \operatorname{span}(f(e_1), f(e_2))$, so $\dim \operatorname{Im} f \leq 2$. In particular, for any $a \in \mathbb{R}$ we have $\dim \operatorname{Im} f \neq \dim \mathbb{R}^3$ and f is never surjective. ■

Exercise 4.3. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that

$$f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} \quad \text{and} \quad f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 14 \\ 18 \end{bmatrix}.$$

(a) Compute the dimensions of $\ker f$ and $\operatorname{Im} f$.

(b) Compute $f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

Solution. We begin from question (b). Since f is linear, if $f(v) = w$ then $f(v/2) = w/2$, so we can compute

$$f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = f\left(\frac{1}{2}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \frac{1}{2} f\left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

This means that we know the images of all the vectors in the standard basis of $\mathbb{R}^3$; these images are the columns of the matrix associated to f with respect to the standard basis:

$$[f] = \begin{bmatrix} 10 & 3 & 2 \\ 14 & 4 & 3 \\ 18 & 5 & 4 \end{bmatrix}.$$

Applying this matrix to the vector $(1, 1, 1)$ we get

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 10 & 3 & 2 \\ 14 & 4 & 3 \\ 18 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10+3+2 \\ 14+4+3 \\ 18+5+4 \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \\ 27 \end{bmatrix}.$$

For question (a), we notice that

$$\begin{bmatrix}3\\4\\5\end{bmatrix} + \begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}5\\7\\9\end{bmatrix} = \frac{1}{2}\begin{bmatrix}10\\14\\18\end{bmatrix},$$

i.e. the three columns of $[f]$ are linearly dependent. Thus the rank of $[f]$ is at most two—in fact, it is exactly 2 because the second and third columns are linearly independent. So we can conclude that $\dim \operatorname{Im} f = 2$ and

$$\dim \ker f = \dim \mathbb{R}^3 - \dim \operatorname{Im} f = 3 - 2 = 1.$$ ■

Exercise 4.4. Let V be the real vector space $\mathbb{R}[x]_{\leq 2}$, whose elements are the polynomials with real coefficients and degree less than or equal to two. Consider the linear map $L\colon V \to V$ that, for any $a, b, c \in \mathbb{R}$, sends the polynomial $ax^2 + bx + c$ to the polynomial $(3a + 3b)x^2 + (b + c)x + (a + b + 2c)$.

(a) Compute the dimensions of the kernel and the image of L.

(b) Decide if L is invertible and, if so, compute the image of the polynomial $4x^2 + x + 1$ through the inverse function.

Solution. The matrix associated to L with respect to the basis $(x^2, x, 1)$ of $\mathbb{R}[x]_{\leq 2}$ is

$$[L] = \begin{bmatrix}3 & 3 & 0\\0 & 1 & 1\\1 & 1 & 2\end{bmatrix}.$$

We begin from question (b): we try to compute the inverse of $[L]$ with the standard algorithm. If we find an inverse, it means that L is invertible, hence bijective. In this case the answer for question (a) follows immediately: $\dim \ker L = 0$ (L is injective) and $\dim \operatorname{Im} L = 3$ (L is surjective).

Recall that the inverse algorithm requires to perform row operations on the matrix

$$\left[\begin{array}{ccc|ccc}3 & 3 & 0 & 1 & 0 & 0\\0 & 1 & 1 & 0 & 1 & 0\\1 & 1 & 2 & 0 & 0 & 1\end{array}\right].$$

It turns out that the inverse matrix does exist and it is

$$\begin{bmatrix}1/6 & -1 & 1/2\\1/6 & 1 & -1/2\\-1/6 & 0 & 1/2\end{bmatrix}.$$

In order to conclude the exercise we have to evaluate the inverse function on the polynomial $4x^2 + x + 1$. It suffices to apply the inverse matrix to the vector

$$\begin{bmatrix}4\\1\\1\end{bmatrix}$$

obtaining

$$\begin{bmatrix} 1/6 \\ 7/6 \\ -1/6 \end{bmatrix},$$

which corresponds to the polynomial

$$\frac{1}{6}x^2 + \frac{7}{6}x - \frac{1}{6}.$$

■

Exercise 4.5. Let $\mathbb{Q}[x]_{\leq 3}$ be the space of polynomials with rational coefficients and degree less than or equal to 3 and let $L\colon \mathbb{Q}[x]_{\leq 3} \to \mathbb{Q}[x]_{\leq 3}$ be the linear map given by

$$L(ax^3 + bx^2 + cx + d) = (a + b + c)x^3 + dx^2 + 2c.$$

Compute a basis of $\ker L$, a basis for $\operatorname{Im} L$, and a basis of $\operatorname{Im}(L \circ L \circ L)$.

Solution. At first we build the matrix associated to L with respect to the standard basis $(x^3, x^2, x, 1)$ of $\mathbb{Q}[x]_{\leq 3}$; for instance, let us compute the first column. We have

$$L(x^3) = (1 + 0 + 0)x^3 + 0x^2 + 2 \cdot 0 = x^3$$

so the first column of $[L]$ is the vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

of the coordinates of x^3 with respect to the standard basis. After the computation of $L(x^2)$, $L(x)$ and $L(1)$ we obtain

$$[L] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

We notice that the rank of $[L]$ is 3 and the pivots are in the first, third and fourth columns, so we can deduce that a basis for $\operatorname{Im} L$ is given by

$$(L(x^3), L(x), L(1)) = (x^3, x^3 + 2, x^2).$$

Now we move to $\ker L$. We know that $\dim \ker L = 4 - \dim \operatorname{Im} L = 1$. A generic polynomial $ax^3 + bx^2 + cx + d \in \mathbb{Q}[x]_{\leq 3}$ belongs to $\ker L$ if and only if

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

that is equivalent to the system

$$\begin{cases} a+b+c=0 \\ d=0 \\ 2c=0. \end{cases}$$

It is easy to show that the set of the solutions of this system is

$$\left\{ \begin{bmatrix} -\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} \;\middle|\; \alpha \in \mathbb{Q} \right\},$$

which is a vector subspace spanned by

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

that corresponds to the polynomial $-x^3 + x^2$. Therefore a basis for $\ker L$ is $(-x^3 + x^2)$.

The last question asks to compute a basis for $\operatorname{Im}(L \circ L \circ L)$. We know that a set of generators is

$$\{(L \circ L \circ L)(x^3), (L \circ L \circ L)(x^2), (L \circ L \circ L)(x), (L \circ L \circ L)(1)\};$$

let us compute these values:

- since $L(x^3) = x^3$, we also have $(L \circ L \circ L)(x^3) = x^3$;
- $(L \circ L \circ L)(x^2) = (L \circ L)(x^3) = x^3$;
- $(L \circ L \circ L)(x) = (L \circ L)(x^3 + 2) = L(L(x^3) + 2L(1)) = L(x^3 + 2x^2) = x^3 + 2x^3 = 3x^3$;
- $(L \circ L \circ L)(1) = (L \circ L)(x^2) = L(x^3) = x^3$.

It follows that

$$\operatorname{Im}(L \circ L \circ L) = \operatorname{span}(x^3, x^3, 3x^3, x^3) = \operatorname{span}(x^3)$$

and a basis for $\operatorname{Im}(L \circ L \circ L)$ is (x^3). ■

Exercise 4.6. Let $L_A \colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map defined as $L_A(v) = Av$, where

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 3 & 2 & -1 \end{bmatrix}.$$

(a) Find a basis of $\ker L_A$ and complete it to a basis of $\mathbb{R}^4$.

(b) Find a basis of $\operatorname{Im} L_A$.

(c) Does the vector $b = \begin{bmatrix} 1 \\ 8 \\ -3 \\ -2 \end{bmatrix}$ belong to $\operatorname{Im} L_A$?

Solution. (a) The row reduced echelon form of A is

$$\tilde{A} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The set of solutions of the system associated to $\tilde{A}$ is

$$\left\{ \begin{bmatrix} z-2t \\ -z+t \\ z \\ t \end{bmatrix} \;\middle|\; z, t \in \mathbb{R} \right\},$$

from which we can obtain a basis of $\ker L_A$:

$$\ker L_A = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right).$$

In order to complete it to a basis of $\mathbb{R}^4$, we row-reduce the matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and we take the columns corresponding to the pivots. It turns out that the pivots are in the first four columns, so a basis of $\mathbb{R}^4$ is

$$\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

(b) The pivots of $\tilde{A}$ are in the first two columns, so a basis of $\operatorname{Im} L_A$ is given by the first two columns of A:

$$\operatorname{Im} L_A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} \right).$$

(c) It is sufficient to row-reduce the matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 2 & 8 \\ 0 & 1 & -3 \\ 1 & 3 & -2 \end{array}\right]$$

and check if there is a pivot in the third column. The reduced form is

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

therefore $b \in \operatorname{Im} L_A$ and, more precisely,

$$b = 7 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}. \qquad \blacksquare$$

Exercise 4.7. Let $\mathbb{K}$ be a field and let $V = \mathbb{K}[x]_{\leq 3}$ be the space of polynomials with degree less than or equal to 3 and coefficients in $\mathbb{K}$. Consider the linear map $f \colon V \to V$ defined as

$$f(ax^3 + bx^2 + cx + d) = (a+b)x^3 + (b+c)x^2 + (c+d)x + (a-d).$$

(a) Compute a basis of $\ker f$, a basis of $\operatorname{Im} f$ and a basis of $\ker f \cap \operatorname{Im} f$ when $\mathbb{K} = \mathbb{R}$.

(b) Compute a basis of $\ker f$, a basis of $\operatorname{Im} f$ and a basis of $\ker f \cap \operatorname{Im} f$ when $\mathbb{K} = \mathbb{Z}_2$.

Solution. We consider the standard basis of V given by $\mathcal{C} = (x^3, x^2, x, 1)$, and find the matrix $[f]$ associated to f with respect to this basis. Recall that the i-th column of $[f]$ is the vector of the coordinates of $f(e_i)$ with respect to $\mathcal{C}$, where e_i is the i-th vector of $\mathcal{C}$.

- $f(x^3) = x^3 + 1$;
- $f(x^2) = x^3 + x^2$;
- $f(x) = x^2 + x$;
- $f(1) = x - 1$.

Therefore

$$[f] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \quad \rightsquigarrow \quad \widetilde{[f]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

where we also row-reduced the matrix.
(a) If $\mathbb{K} = \mathbb{R}$, we have that $\text{rk}[f] = 4$, so $\ker f = \{0\}$, $\text{Im}\, f = V$ and $\ker f \cap \text{Im}\, f = \{0\}$.
(b) If $\mathbb{K} = \mathbb{Z}_2$, $2 = 0$, so $[f]$ has rank 3 and the vectors corresponding to the columns of the pivots of $\widetilde{[f]}$ form a basis of $\text{Im}\, f$. In this case

$$\text{Im}\, f = \text{span}\left(f(x^3), f(x^2), f(x)\right) = \text{span}(x^3 + 1, x^3 + x^2, x^2 + x).$$

In order to compute a basis of $\ker f$, we solve the system

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{that is} \quad \begin{cases} x + y = 0 \\ y + z = 0 \\ z + t = 0. \end{cases}$$

Recalling that $-1 = 1$ in $\mathbb{Z}_2$, we easily get that $x = y = z = t$ is a solution, thus

$$\ker[f] = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right),$$

from which we can recover a basis of $\ker f$:

$$\ker f = \text{span}(x^3 + x^2 + x + 1).$$

Now, since $\dim \ker f = 1$, we have that either $\ker f \cap \text{Im}\, f = \{0\}$ or $\ker f \cap \text{Im}\, f = \ker f$, and the latter is true if and only if $x^3 + x^2 + x + 1 \in \text{Im}\, f$. But it is easy to show that

$$x^3 + x^2 + x + 1 = (x^3 + 1) + (x^2 + x) = f(x^3) + f(x) \in \text{Im}\, f,$$

so we can conclude $\ker f \cap \text{Im}\, f = \ker f$ and a basis is given by $(x^3 + x^2 + x + 1)$. ■

Exercise 4.8. Let $f\colon (\mathbb{Z}_7)^4 \to (\mathbb{Z}_7)^3$ be the linear map associated to the matrix

$$A = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 2 & 0 & 1 \\ 1 & 1 & 3 & -1 \end{bmatrix}$$

with respect to the standard basis. Find a basis of $\ker f$ and a basis of $\text{Im}\, f$.

Solution. We begin reducing A to its row echelon form. Remember that the coefficients lie in $\mathbb{Z}_7$, so all the operations have to be performed mod 7.

$$\begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 2 & 0 & 1 \\ 1 & 1 & 3 & -1 \end{bmatrix}$$

$$\Big\downarrow \ \text{(second row)} - 3\text{(first row)}, \ \text{(third row)} - \text{(first row)}$$

$$\begin{bmatrix} 1 & 2 & 5 & 2 \\ 0 & 3 & 6 & 2 \\ 0 & -1 & -2 & -3 \end{bmatrix}$$

$$\Big\downarrow \ -2\text{(second row)}$$

$$\begin{bmatrix} 1 & 2 & 5 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{bmatrix}$$

$$\Big\downarrow \ \text{(third row)} + \text{(second row)}$$

$$\begin{bmatrix} 1 & 2 & 5 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivots are in the first two columns, so these columns of A are a basis of the image of f:

$$\operatorname{Im} f = \operatorname{span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right).$$

For $\ker f$, we solve the system associated to the row reduced form of A:

$$\begin{cases} x + 2y + 5z + 2t = 0 \\ y + 2z + 3t = 0. \end{cases}$$

We choose z and t as free variables and obtain

$$\ker f = \left\{ \begin{bmatrix} -z - 3t \\ -2z - 3t \\ z \\ t \end{bmatrix} \middle| \ z, t \in \mathbb{Z}_7 \right\} = \operatorname{span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right). \qquad \blacksquare$$

Exercise 4.9. Let $\mathbb{K}$ be a field and let $T\colon \mathbb{K}^3 \to \mathbb{K}^3$ be the linear map associated to the matrix

$$[T] = \begin{bmatrix} 9 & -1 & 7 \\ -1 & 5 & 5 \\ 1 & -1 & 3 \end{bmatrix}$$

with respect to the standard basis.

(a) Determine $\ker T$ and $\operatorname{Im} T$ when $\mathbb{K} = \mathbb{Z}_2$.

(b) Determine $\ker T$ and $\operatorname{Im} T$ when $\mathbb{K} = \mathbb{Z}_3$.

(c) In which of the previous cases is it true that $\mathbb{K}^3 = \ker T \oplus \operatorname{Im} T$?

Exercise 4.10. Let

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 2 & 1 \\ 0 & 5 \end{bmatrix} \in \mathcal{M}_{4\times 2}(\mathbb{R})$$

and let v_1, v_2 be its two columns.

(a) Find $\operatorname{Im} V$ and $\ker V$.

(b) For which values of $a \in \mathbb{R}$ does the vector $w = \begin{bmatrix} 1 \\ a \\ a \\ a \end{bmatrix}$ belong to $\operatorname{Im} V$?

(c) Find two vectors $v_3, v_4 \in \mathbb{R}^4$ such that (v_1, v_2, v_3, v_4) is a basis of $\mathbb{R}^4$.

Solution. (a) The two vectors v_1 and v_2 are linearly independent. This can be seen by row-reducing V, or by noticing that they are not multiple of each other. Thus (v_1, v_2) is a basis of $\operatorname{Im} V$ and its dimension is 2. It follows that $\dim \ker V = \dim \mathbb{R}^2 - \dim \operatorname{Im} V = 0$, so $\ker V = \{0\}$.
(b) The vector w belongs to $\operatorname{Im} V$ if and only if the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & a \\ 2 & 1 & a \\ 0 & 5 & a \end{bmatrix}$$

has not rank 3. Its row reduced form is

$$\begin{bmatrix} 1 & 3 & a \\ 0 & 1 & 1 \\ 0 & 0 & -a+5 \\ 0 & 0 & a-5 \end{bmatrix},$$

so $w \in \operatorname{Im} V$ if and only if $a = 5$.

(c) One possible method to find v_3 and v_4 is to reduce the matrix

$$\left[\begin{array}{cc|cccc} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

and take the columns given by the pivots. After a quick computation we get

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{bmatrix},$$

therefore a basis of $\mathbb{R}^4$ is

$$\left(v_1, v_2, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right). \qquad \blacksquare$$

Exercise 4.11. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map and let

$$A = [f] = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 1 & -4 & 5 \end{bmatrix}$$

be the matrix associated to f with respect to the standard basis.

(a) Does the vector $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ belong to $\operatorname{Im} f$?

(b) Consider the vectors

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Prove that $\mathcal{B} = (v_1, v_2, v_3)$ is a basis of $\mathbb{R}^3$.

(c) Write the matrix $[f]_{\mathcal{B}}^{\mathcal{B}}$ associated to f with respect to the basis $\mathcal{B}$ both in input and in output.

Solution. (a) The vector v belongs to $\operatorname{Im} f$ if and only if it belongs to the span of the columns of the matrix A, and this happens if and only if the rank of A is equal to the rank of the complete matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 3 & 0 & 2 \\ 1 & -4 & 5 & 3 \end{array}\right].$$

By using simple row operations we find that both ranks are equal to 3, so v belongs to $\operatorname{Im} f$.
(b) Three vectors in a three-dimensional vector space form a basis if and only if they are linearly independent. To check this for v_1, v_2, v_3, we put them in a matrix

$$B = \left[v_1 \middle| v_2 \middle| v_3 \right] = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

and compute its rank, for example by finding its row reduced echelon form. It turns out that this rank is 3, so v_1, v_2 and v_3 are linearly independent.
(c) The matrix B computed above is actually a basis change matrix from $\mathcal{B}$ to the standard basis $\mathcal{C} = (e_1, e_2, e_3)$, i.e. it is the matrix $[\mathrm{id}]^{\mathcal{C}}_{\mathcal{B}}$. To compute the matrix associated to f with respect to $\mathcal{B}$, we have the formula

$$[f]^{\mathcal{B}}_{\mathcal{B}} = [\mathrm{id}]^{\mathcal{B}}_{\mathcal{C}} [f] [\mathrm{id}]^{\mathcal{C}}_{\mathcal{B}} = B^{-1} A B.$$

We have to find B^{-1} with the usual algorithm:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & -2 \end{array}\right]$$

and finally obtain

$$[f]^{\mathcal{B}}_{\mathcal{B}} = B^{-1} A B = \begin{bmatrix} 8 & 2 & -2 \\ 2 & 3 & -1 \\ 11 & 2 & -2 \end{bmatrix}. \qquad \blacksquare$$

Exercise 4.12. Consider the following two ordered subsets of $\mathbb{R}^3$:

$$\mathcal{C} = \left(\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right), \quad \mathcal{B} = \left(\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x+y \\ x+z \\ x+z \end{bmatrix}.$$

Find a basis of $\ker L$, a basis of $\operatorname{Im} L$, and write the matrix $[L]^{\mathcal{B}}_{\mathcal{C}}$ associated to L with respect to the basis $\mathcal{C}$ in input and to the basis $\mathcal{B}$ in output.

Solution. The matrix associated to L with respect to the standard basis $\mathcal{E} = (e_1, e_2, e_3)$ is

$$A = [L]^{\mathcal{E}}_{\mathcal{E}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and the row reduced echelon form is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{$*$}$$

therefore a basis for $\operatorname{Im} L$ is given by the columns where the pivots are, that is

$$\operatorname{Im} L = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

A basis for $\ker L$ can be found solving the system obtained by ($*$):

$$\begin{cases} x + y = 0 \\ -y + z = 0, \end{cases}$$

for which we choose e.g. z as a free variable and get $x = -z$, $y = z$. In conclusion

$$\ker L = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right).$$

In order to answer the last question, we need the basis change matrices for $\mathcal{B}$ and $\mathcal{C}$. In fact, we have

$$[L]_{\mathcal{C}}^{\mathcal{B}} = [\mathrm{id}]_{\mathcal{E}}^{\mathcal{B}} [L]_{\mathcal{E}}^{\mathcal{E}} [\mathrm{id}]_{\mathcal{C}}^{\mathcal{E}}$$

or, graphically,

$$\mathbb{R}^3(\mathcal{C}) \xrightarrow{\ \mathrm{id}\ } \mathbb{R}^3(\mathcal{E}) \xrightarrow{\ L\ } \mathbb{R}^3(\mathcal{E}) \xrightarrow{\ \mathrm{id}\ } \mathbb{R}^3(\mathcal{B})$$

The vectors of the sets $\mathcal{B}$ and $\mathcal{C}$ are already written with respect of the basis $\mathcal{E}$, so we can easily get

$$[\mathrm{id}]_{\mathcal{C}}^{\mathcal{E}} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 3 & 3 & 1 \end{bmatrix} \quad \text{and} \quad [\mathrm{id}]_{\mathcal{B}}^{\mathcal{E}} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The only thing to do is to compute $[\mathrm{id}]_{\mathcal{E}}^{\mathcal{B}} = ([\mathrm{id}]_{\mathcal{B}}^{\mathcal{E}})^{-1}$ with the usual algorithm

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & -1/3 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

and finally

$$[L]_{\mathcal{C}}^{\mathcal{B}} = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5/3 & 1 & 0 \\ 10/3 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \blacksquare$$

Exercise 4.13. Consider the linear map $\Phi\colon \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ such that, for any matrix $A \in \mathcal{M}_2(\mathbb{R})$, we have $\Phi(A) = AB$, where B is the matrix

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

(a) Determine the dimension and a basis of $\ker \Phi$.

(b) Determine the dimension and a basis of $\operatorname{Im} \Phi$.

Solution. Let $\mathcal{C} = (E_1, E_2, E_3, E_4)$ be the standard basis of $\mathcal{M}_2(\mathbb{R})$, i.e.

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix $[\Phi]$ associated to Φ with respect to the standard basis is a 4×4 matrix such that the i-th column is the coordinate vector (with respect to the basis $\mathcal{C}$) of the image $\Phi(E_i)$ of the i-th vector of $\mathcal{C}$.

$$\begin{aligned}
\Phi(E_1) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = E_1 + 2E_2; \\
\Phi(E_2) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} = 2E_1 + 4E_2; \\
\Phi(E_3) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = E_3 + 2E_4; \\
\Phi(E_4) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix} = 2E_3 + 4E_4;
\end{aligned}$$

therefore

$$[\Phi] = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix},$$

which can be reduced via the usual algorithm to the form

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that

$$\operatorname{Im}[\Phi] = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right) \quad \Rightarrow \quad \operatorname{Im} \Phi = \operatorname{span}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \right)$$

and its dimension is 2, whereas to compute $\ker[\Phi]$ we have to solve the system

$$\begin{cases} x+2y=0 \\ z+2t=0, \end{cases}$$

from which we obtain

$$\ker[\Phi] = \operatorname{span}\left(\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-2\\1\end{bmatrix}\right) \quad \Rightarrow \quad \ker\Phi = \operatorname{span}\left(\begin{bmatrix}-2 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\-2 & 1\end{bmatrix}\right)$$

and its dimension is 2. ■

Exercise 4.14. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined as

$$f\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x-y\\x+2y+3z\\y+z\end{bmatrix}.$$

(a) Compute a basis of $\ker f$ and a basis of $\operatorname{Im} f$, and determine if these two subspaces are in direct sum.

(b) Find, if possible, a non-zero linear map $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $g \circ f$ is the zero map.

Solution. (a) The matrix associated to f with respect to the standard basis is

$$A = \begin{bmatrix}1 & -1 & 0\\1 & 2 & 3\\0 & 1 & 1\end{bmatrix}.$$

In order to compute a basis of $\operatorname{Im} f$, we reduce A in its row echelon form, obtaining

$$\tilde{A} = \begin{bmatrix}1 & -1 & 0\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}.$$

Thus $\dim \operatorname{Im} f = \operatorname{rk} A = 2$ and a basis is given by the first and second columns of A (corresponding to the pivots of $\tilde{A}$):

$$\operatorname{Im} f = \operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}\right).$$

In order to find a basis of $\ker f$, we solve the system

$$\begin{bmatrix}1 & -1 & 0\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

whose set of solutions is

$$\ker f = \left\{ \begin{bmatrix} -\lambda \\ -\lambda \\ \lambda \end{bmatrix} \;\middle|\; \lambda \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right).$$

The two subspaces $\ker f$ and $\operatorname{Im} f$ are in direct sum if and only if $\ker f \cap \operatorname{Im} f = \{0\}$, which is equivalent to $\ker f \not\subseteq \operatorname{Im} f$ (because $\dim \ker f = 1$). Therefore

$$\ker f \cap \operatorname{Im} f = \{0\} \quad \text{if and only if} \quad \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \notin \operatorname{Im} f.$$

We know that $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \in \operatorname{Im} f$ if and only if there exist $a, b \in \mathbb{R}$ such that

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

that is to say, the system

$$\begin{cases} a - b = -1 \\ a + 2b = -1 \\ b = 1 \end{cases}$$

has a solution. But it is easy to see that the first two equations are incompatible, thus $\ker f$ and $\operatorname{Im} f$ are in direct sum.

(b) In order to define a linear map, it is sufficient to say what the images of the vectors of a basis are; the image of a generic vector is then uniquely determined by linearity. We choose the basis $\mathcal{B} = (v_1, v_2, v_3)$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

i.e. (v_1, v_2) is a basis of $\operatorname{Im} f$ and (v_3) is a basis of $\ker f$. Notice that $\mathcal{B}$ is a basis of the whole space $\mathbb{R}^3$ because in (a) we proved that $\mathbb{R}^3 = \operatorname{Im} f \oplus \ker f$. The map g defined as

$$g(v_1) = 0, \quad g(v_2) = 0, \quad g(v_3) = v_3$$

is not the zero map, but $g(w) = 0$ for any $w \in \operatorname{Im} f$, therefore $(g \circ f)(v) = g(f(v)) = 0$ for any $v \in \mathbb{R}^3$, i.e. $g \circ f$ is the zero map. ∎

Exercise 4.15. Let $T\colon \mathbb{Q}[x]_{\leq 3} \to \mathbb{Q}^3$ be the linear map given by

$$T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a + b \\ c + d \\ a + b + c + d \end{bmatrix}.$$

(a) Compute a basis of $\ker T$ and a basis of $\operatorname{Im} T$.

(b) Define a linear map $F\colon \mathbb{Q}^3 \to \mathbb{Q}[x]_{\leq 3}$ such that $\ker T \oplus \operatorname{Im} F = \mathbb{Q}[x]_{\leq 3}$, if such a map exists.

Solution. (a) We choose the basis $(x^3, x^2, x, 1)$ for $\mathbb{Q}[x]_{\leq 3}$ and the standard basis for $\mathbb{Q}^3$. With respect to these two bases, the matrix $[T]$ associated to T is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Since $\operatorname{Im} T$ is the span of the columns of $[T]$, the two linearly independent columns

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

form a basis of $\operatorname{Im} T$.

In order to compute $\ker T$, we reduce $[T]$ to the row echelon form, obtaining

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore a polynomial $ax^3 + bx^2 + cx + d$ belongs to $\ker T$ if and only if

$$\begin{cases} a + b = 0 \\ c + d = 0. \end{cases}$$

We can choose (for example) b and d freely, and the other coefficients are forced to be $a = -b$ and $c = -d$. So a polynomial belongs to $\ker T$ if and only if its coefficients have the form $(-b, b, -d, d)$ for some $b, d \in \mathbb{Q}$, which we may write as

$$b \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

The polynomials corresponding to these two vectors, i.e. $-x^3 + x^2$ and $-x + 1$, form a basis of $\ker T$.

(b) In order to answer the question, at first we find a subspace $V \subseteq \mathbb{Q}[x]_{\leq 3}$ such that $\ker T \oplus V = \mathbb{Q}[x]_{\leq 3}$, then we define a linear function $F\colon \mathbb{Q}^3 \to \mathbb{Q}[x]_{\leq 3}$ such that $\operatorname{Im} F = V$.

Having fixed a basis, we may think of $\mathbb{Q}[x]_{\leq 3}$ as $\mathbb{Q}^4$, exchanging a polynomial with the 4-tuple of its coefficients. For answer (a), we saw that

$$\ker T = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

(more precisely, it is the span of the polynomials that have these 4-tuples as coefficients). We only have to find other two linearly independent vectors of $\mathbb{Q}^4$ such that these four vectors form a basis of $\mathbb{Q}^4$. One possible choice is

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix},$$

corresponding to polynomials x^3 and 1 respectively. So, we may define $V = \operatorname{span}(x^3, 1)$. Now, if (e_1, e_2, e_3) is the standard basis of $\mathbb{Q}^3$, the function $F\colon \mathbb{Q}^3 \to \mathbb{Q}[x]_{\leq 3}$ such that $F(e_1) = x^3$, $F(e_2) = 1$, $F(e_3) = 0$ (and extended by linearity) satisfies the required conditions. ■

Exercise 4.16. Let $L\colon \mathbb{Q}^4 \to \mathbb{Q}[x]_{\leq 3}$ the linear map defined by

$$L\left(\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right) = (a+b)x^3 + (c+d)x^2 + 2cx + 2d.$$

(a) Find a basis of $\ker L$ and a basis of $\operatorname{Im} L$.

(b) Determine if exists a linear map $G\colon \mathbb{Q}[x]_{\leq 3} \to \mathbb{Q}^4$ such that $\ker G = \operatorname{Im} L$ and $\operatorname{Im} G = \ker L$.

Chapter 5

Eigenvalues and Eigenvectors

Exercise 5.1. Let M be a 2×2 matrix with real coefficients and eigenvalues 3 and 5, with eigenvectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ respectively.

(a) Compute $M \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

(b) Find a diagonal matrix D and two matrices A, A^{-1} (each inverse of the other) such that $M = ADA^{-1}$.

Exercise 5.2. Let $M \in \mathcal{M}_3(\mathbb{R})$ be the matrix

$$M = \begin{bmatrix} -2 & 0 & 1 \\ -2 & 0 & 1 \\ -4 & 0 & 2 \end{bmatrix}.$$

Compute the eigenvalues and the eigenvectors of M. Is M diagonalizable?

Solution. The characteristic polynomial is

$$p_M(\lambda) = \det \begin{bmatrix} -2-\lambda & 0 & 1 \\ -2 & -\lambda & 1 \\ -4 & 0 & 2-\lambda \end{bmatrix} = -\lambda^3$$

(use Laplace expansion on the second column), so the only eigenvalue is 0 with algebraic multiplicity $m_0 = 3$. Let us compute its geometric multiplicity:

$$\ker(M - 0 \cdot I) = \ker M = \ker \begin{bmatrix} -2 & 0 & 1 \\ -2 & 0 & 1 \\ -4 & 0 & 2 \end{bmatrix} = \ker \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If we use x_2 and x_3 as free variables, we obtain

$$\ker M = \left\{ \begin{bmatrix} x_3/2 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_2, x_3 \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

and that is also a basis of $\ker M$. It follows that $g_0 = \dim\ker M = 2$ and M is not diagonalizable. Moreover, since 0 is the only eigenvalue, the eigenvectors of M are all and only the vectors of $\ker M$. ■

Exercise 5.3. Let $M \in \mathcal{M}_3(\mathbb{R})$ be the matrix

$$M = \begin{bmatrix} -2 & 2 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Compute the eigenvalues and the eigenvectors of M. Is M diagonalizable?

Solution. The characteristic polynomial of M is

$$p_M(\lambda) = \det \begin{bmatrix} -2-\lambda & 2 & 2 \\ -1 & 1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{bmatrix} = -\lambda^3.$$

The only eigenvalue is 0 with algebraic multiplicity $m_0 = 3$. We compute the relative eigenspace:

$$V_0 = \ker M = \ker \begin{bmatrix} -2 & 2 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \ker \begin{bmatrix} -2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We let y and z be the free variables and, from the first equation of the last system, deduce $x = y + z$. Therefore

$$V_0 = \left\{ \begin{bmatrix} y+z \\ y \\ z \end{bmatrix} \middle|\, y, z \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right).$$

The non-zero vectors in V_0 are the eigenvectors of M. Since $g_0 = \dim V_0 = 2 \neq m_0$, M is not diagonalizable. ■

Exercise 5.4. Find a basis of $\mathbb{R}^3$ made of eigenvectors for the linear map $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ that is represented by the matrix

$$[L] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

with respect to the standard basis.

Solution. The characteristic polynomial of L is

$$p_L(\lambda) = \det \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^3 + 1 + 1 - 3(2-\lambda) =$$
$$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4.$$

We notice that $\lambda = 1$ is a solution and proceed with the polynomial division:

$$\frac{-\lambda^3 + 6\lambda^2 - 9\lambda + 4}{\lambda - 1} = -\lambda^2 + 5\lambda - 4 = (\lambda - 1)(4 - \lambda).$$

Therefore the eigenvalues of L are 1, with algebraic multiplicity $m_1 = 2$, and 4, with algebraic multiplicity $m_4 = 2$. Now we look for eigenvectors.

- For $\lambda = 1$, we have

$$\ker([L] - I) = \ker \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$= \left\{ \begin{bmatrix} -y - z \\ y \\ z \end{bmatrix} \,\middle|\, y, z \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

- For $\lambda = 4$, we have

$$\ker([L] - 4I) = \ker \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

We row-reduce the matrix:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\downarrow \text{ permute rows}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\downarrow \begin{matrix} \text{(second row)} - \text{(first row)} \\ \text{(third row)} + 2\text{(first row)} \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\downarrow \text{ (third row)} + \text{(second row)}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

If we keep z as a free variable, from the second equation we get $y = z$ and then substituting into the first one we obtain $x + z - 2z = 0$ i.e. $x = z$. Therefore

$$\ker([L] - 4I) = \left\{ \begin{bmatrix} z \\ z \\ z \end{bmatrix} \,\middle|\, z \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right).$$

The vectors

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

are linearly independent and form a basis of $\mathbb{R}^3$ made of eigenvectors for L. ■

Exercise 5.5. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the endomorphism associated to the matrix

$$\begin{bmatrix} 9 & 6 & 9 \\ 3 & 12 & 9 \\ 3 & 6 & 15 \end{bmatrix}$$

with respect to the standard basis. Say if T is diagonalizable and, if so, compute a basis of $\mathbb{R}^3$ made of eigenvectors for T.

Solution. In order to simplify the computation, we will work with the matrix

$$A = \frac{1}{3}[T] = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 4 & 3 \\ 1 & 2 & 5 \end{bmatrix},$$

remembering that A is diagonalizable if and only if $[T]$ is diagonalizable. Moreover, if λ is an eigenvalue for $[T]$ with eigenvector v, then $\lambda/3$ is an eigenvalue for A with the same eigenvector.

The characteristic polynomial of A is

$$p_A(\lambda) = \det \begin{bmatrix} 3-\lambda & 2 & 3 \\ 1 & 4-\lambda & 3 \\ 1 & 2 & 5-\lambda \end{bmatrix} = -\lambda^3 + 12\lambda^2 - 36\lambda + 32,$$

which has roots 2 and 8 with multiplicities $m_2 = 2$ and $m_8 = 1$ respectively. To check if A is diagonalizable we have to compute the geometric multiplicity g_2 of the eigenvalue 2 (we already know that the geometric multiplicity of 8 is $g_8 = m_8 = 1$, because $1 \leq g_8 \leq m_8 = 1$). We have

$$A - 2I = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

that has rank 1, hence $g_2 = \dim\ker(A-2I) = 2$ which is equal to m_2. Therefore A is diagonalizable, and so $[T]$ is diagonalizable. Let us find the eigenvectors: for the eigenvalue 2, we have to solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

that is to say $x + 2y + 3z = 0$. We can let y, z be free variables and obtain

$$\ker(A-2I) = \left\{ \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} \;\middle|\; y, z \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right).$$

Now we have to compute $\ker(A-8I)$, i.e. we have to solve

$$\begin{bmatrix} -5 & 2 & 3 \\ 1 & -4 & 3 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After a little computation we obtain

$$\ker(A-8I) = \left\{ \begin{bmatrix} z \\ z \\ z \end{bmatrix} \;\middle|\; z \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right).$$

In conclusion a basis of $\mathbb{R}^3$ made of eigenvectors of T is

$$\mathcal{B} = \left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right).$$

The matrix associated to T with respect to $\mathcal{B}$ is

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 24 \end{bmatrix}.$$

■

Exercise 5.6. Let

$$M = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 0 \end{bmatrix} \in \mathcal{M}_3(\mathbb{R}).$$

(a) Determine the eigenvalues and the eigenvectors of M over the field $\mathbb{R}$.

(b) Say, justifying the answer, if there exist a matrix $V \in \mathcal{M}_3(\mathbb{R})$ and a diagonal matrix $D \in \mathcal{M}_3(\mathbb{R})$ such that $M = VDV^{-1}$.

(c) Determine a possible choice of such V and D.

Solution. (a) The characteristic polynomial of M is

$$p_M(\lambda) = \det \begin{bmatrix} -\lambda & 2 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 2 & -\lambda \end{bmatrix} = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = (-1-\lambda)^2(5-\lambda).$$

The eigenvalues of M are the roots of $p_M(\lambda)$, i.e. -1 and 5. Let us find the eigenvectors. The eigenspace relative to -1 is

$$V_{-1} = \ker(M+I) = \ker \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and a quick computation gives

$$V_{-1} = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

The eigenspace relative to 5 is

$$V_5 = \ker(M-5I) = \ker \begin{bmatrix} -5 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -5 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

where we row-reduced the matrix $M-5I$ as usual. The solutions of the associated system are

$$V_5 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right).$$

(b) From answer (a) we have that $m_{-1} = g_{-1} = 2$ and $m_5 = g_5 = 1$, so M is diagonalizable and the matrices V and D do exist. (Otherwise, notice that M is symmetric, thus diagonalizable.)
(c) The matrices V and D are respectively the matrix of the eigenvectors and the matrix of the eigenvalues of M: if we put

$$V = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

it is easy to show that $M = VDV^{-1}$. ■

Exercise 5.7. Let A be the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues and the eigenvectors of A over the field $\mathbb{R}$. Is A diagonalizable?

(b) Find the eigenvalues and the eigenvectors of A over the field $\mathbb{C}$. Is A diagonalizable?

(c) Find an invertible matrix $V \in \mathcal{M}_3(\mathbb{C})$ such that $V^{-1}AV$ is diagonal.

Solution. (a) The characteristic polynomial of A is

$$p_A(\lambda) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\big((2-\lambda)^2+1\big).$$

Over the field $\mathbb{R}$, the only eigenvalue is $\lambda = 1$, because $(2-\lambda)^2 + 1 > 0$ for all $\lambda \in \mathbb{R}$, so that polynomial has no roots in $\mathbb{R}$. It follows immediately that A is not diagonalizable over $\mathbb{R}$, since not all the roots of p_A belong to $\mathbb{R}$.

The eigenspace associated to the eigenvalue 1 is

$$V_1 = \ker(A-I) = \ker \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is $\operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

(b) The real eigenvalue and the relative eigenvectors found in (a) are still present in the complex case. We have to add the roots of $(2-\lambda)^2+1$, that are

$$2-\lambda = \pm i \quad \rightsquigarrow \quad \lambda = 2 \mp i.$$

Since the three eigenvalues are different, we conclude that A is diagonalizable over $\mathbb{C}$.

- The eigenspace relative to the eigenvalue $2+i$ is

$$V_{2+i} = \ker(A-(2+i)I) = \ker \begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & -1-i \end{bmatrix} \overset{(\star)}{=} \ker \begin{bmatrix} -i & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1-i \end{bmatrix}$$

where in $(\star)$ we added the first row multiplied by i to the second. The system associated to the last matrix is

$$\begin{cases} -ix + y = 0 \\ (-1-i)z = 0 \end{cases}$$

and, if x is a free variable, we have

$$V_{2+i} = \left\{ \begin{bmatrix} x \\ ix \\ 0 \end{bmatrix} \;\middle|\; x \in \mathbb{C} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}\right).$$

- The eigenspace relative to the eigenvalue $2-i$ is

$$V_{2-i} = \ker(A-(2-i)I) = \ker \begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & -1+i \end{bmatrix} \overset{(\star)}{=} \ker \begin{bmatrix} i & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1+i \end{bmatrix}$$

where in $(\star)$ we added the first row multiplied by $-i$ to the second. The system associated to the last matrix is

$$\begin{cases} ix+y=0 \\ (-1+i)z=0 \end{cases}$$

and, if x is a free variable, we have

$$V_{2-i} = \left\{ \begin{bmatrix} x \\ -ix \\ 0 \end{bmatrix} \middle| \, x \in \mathbb{C} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \right).$$

(c) It is sufficient to define V as a matrix that has three linearly independent eigenvectors as columns:

$$V = \begin{bmatrix} 0 & 1 & 1 \\ 0 & i & -i \\ 1 & 0 & 0 \end{bmatrix}.$$

We know that in this case

$$V^{-1}AV = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}.$$

■

Exercise 5.8. Let

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 4 & 3 \\ 1 & 2 & 5 \end{bmatrix}.$$

(a) Is A diagonalizable over $\mathbb{R}$?

(b) Is A diagonalizable over $\mathbb{Z}_3$?

Solution. (a) The characteristic polynomial of A is

$$p_A(\lambda) = \det \begin{bmatrix} 3-\lambda & 2 & 3 \\ 1 & 4-\lambda & 3 \\ 1 & 2 & 5-\lambda \end{bmatrix} = -\lambda^3 + 12\lambda^2 - 36\lambda + 32.$$

At first we look for rational roots of $p_A(\lambda)$. Recall that, if a polynomial $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ with integer coefficients has a rational root p/q (with p and q coprime), then p divides a_0 and q divides a_n. So, if α is a rational root of $p_A(\lambda)$, then α must belong to the set $\{\pm1, \pm2, \pm4, \pm8, \pm16, \pm32\}$ of the divisors of 32. We try them one by one, until we find that $p_A(2) = 0$. Via the Ruffini algorithm we compute

$$p_A(\lambda) = (\lambda - 2)(-\lambda^2 + 10\lambda - 16)$$

and then, finally,

$$p_A(\lambda) = -(\lambda - 8)(\lambda - 2)^2.$$

This means that the eigenvalues of A are: 2, with algebraic multiplicity $m_2 = 2$, and 8, with algebraic multiplicity $m_8 = 1$. To check if A is diagonalizable over $\mathbb{R}$ we have to compute the geometric multiplicity g_2 of the eigenvalue 2 (we already know that the geometric multiplicity of 8 is $g_8 = m_8 = 1$, because $1 \le g_8 \le m_8 = 1$). We have

$$A - 2I = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

that has rank 1, hence $g_2 = \dim\ker(A - 2I) = 2$ which is equal to m_2. Therefore A is diagonalizable over $\mathbb{R}$.
(b) When read over $\mathbb{Z}_3$, the matrix A becomes

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

and the characteristic polynomial is

$$p_A(\lambda) = (\lambda + 1)^3.$$

The only eigenvalue is -1 with algebraic multiplicity $m_{-1} = 3$. Let's check the geometric multiplicity: the matrix

$$A + I = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

has rank 1, so the geometric multiplicity of -1 is $g_{-1} = 2 \ne m_{-1}$. We deduce that A is *not* diagonalizable over $\mathbb{Z}_3$. ■

Exercise 5.9. Let

$$M = \begin{bmatrix} 3 & 3 & 2 \\ 0 & 1 & 0 \\ 2 & -2 & 3 \end{bmatrix}.$$

(a) Compute the eigenvalues and the eigenvectors of M over the field $\mathbb{R}$.

(b) Is M diagonalizable over $\mathbb{R}$?

(c) Is M diagonalizable over $\mathbb{Z}_5$?

Exercise 5.10. Let $\mathbb{K}$ be a field and let $T\colon \mathbb{K}^4 \to \mathbb{K}^4$ be the endomorphism whose associated matrix (with respect to the standard basis) is

$$[T] = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of T.

(b) Is T diagonalizable for $\mathbb{K} = \mathbb{R}$? For $\mathbb{K} = \mathbb{C}$? For $\mathbb{K} = \mathbb{Z}_5$?

Exercise 5.11. Let $a \in \mathbb{R}$ and consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & a \end{bmatrix}.$$

(a) For which $a \in \mathbb{R}$ is A diagonalizable over $\mathbb{R}$?

(b) For which $a \in \mathbb{R}$ is $\lambda = 3$ an eigenvalue for A?

(c) For which $a \in \mathbb{R}$ is $v = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ an eigenvector for A?

Solution. The characteristic polynomial of A is

$$p_A(\lambda) = \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & a-\lambda \end{bmatrix} = \lambda^2 - (a+1)\lambda + (a+1).$$

(a) A is diagonalizable over $\mathbb{R}$ if and only if $p_A(\lambda)$ splits through linear factors over $\mathbb{R}$, and the algebraic and geometric multiplicities of the roots are the same. Let us find the roots: the discriminant of $p_A(\lambda)$ is $\Delta = (a+1)^2 - 4(a+1) = (a+1)(a-3)$, from which we have

$$\lambda_1 = \frac{a+1+\sqrt{\Delta}}{2} \quad \text{and} \quad \lambda_2 = \frac{a+1-\sqrt{\Delta}}{2}.$$

- If $\Delta > 0$, that is $a < -1$ or $a > 3$, $p_A(\lambda)$ has two distinct real roots (that have algebraic multiplicity 1), so A is diagonalizable.
- If $\Delta < 0$, that is $-1 < a < 3$, $p_A(\lambda)$ is irreducible over $\mathbb{R}$, so A is not diagonalizable over $\mathbb{R}$.
- If $\Delta = 0$, that is $a = -1$ or $a = 3$, we have only one eigenvalue $\lambda = (a+1)/2$ with algebraic multiplicity $m_\lambda = 2$. We have to compute the geometric multiplicity.

– For $a = -1$, the eigenvalue is $\lambda = 0$ and the matrix becomes

$$A - \lambda I = A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

which clearly has rank 1. Therefore $g_\lambda = \dim\ker(A - \lambda I) = 2 - 1 = 1 < m_\lambda$ and A is not diagonalizable.

– For $a = 3$, the eigenvalue is $\lambda = 2$ and the matrix becomes

$$A - \lambda I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

and also in this case its rank is 1, so A is not diagonalizable.

(b) $\lambda = 3$ is an eigenvalue for A if and only if it is a root of $p_A(\lambda)$, that is to say $p_A(3) = 0$. But

$$p_A(3) = 3^2 - (a+1) \cdot 3 + (a+1) = 7 - 2a$$

and it is 0 if and only if $a = 7/2$.
(c) The vector v is an eigenvector for A if and only if there exists $\lambda \in \mathbb{R}$ such that $Av = \lambda v$, that is

$$\begin{bmatrix} 1 & -1 \\ 1 & a \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \lambda \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

This brings to the system

$$\begin{cases} 5 = 4\lambda \\ 4 - a = -\lambda, \end{cases}$$

which has a unique solution $\lambda = 5/4$ and $a = 21/4$. Therefore v is an eigenvector for A if and only if $a = 21/4$ (and the relative eigenvalue is $\lambda = 5/4$). ■

Exercise 5.12. Let $k \in \mathbb{R}$ and let $B_k \colon \mathbb{R}^3 \to \mathbb{R}^3$ be an endomorphism represented by the matrix

$$[B_k] = \begin{bmatrix} 2 & 0 & 0 \\ k & 1 & 0 \\ 5 & k-2 & 1 \end{bmatrix}$$

with respect to the standard basis of $\mathbb{R}^3$.

(a) Find the values of k for which the endomorphism B_k is diagonalizable.

(b) Let $k = 3$. Is it true or false that there exists a basis of $\mathbb{R}^3$ such that the matrix associated to B_3 with respect to this basis is

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}?$$

Solution. (a) The characteristic polynomial of B_k is $p(\lambda) = (2-\lambda)(1-\lambda)^2$ for any k. The eigenvalues are $\lambda = 2$ with algebraic multiplicity $m_2 = 1$ and $\lambda = 1$ with algebraic multiplicity $m_1 = 2$. We know that the matrix is diagonalizable if these multiplicities are equal to the dimensions of the relative eigenspaces $V_2 = \ker(B_k - 2I)$ and $V_1 = \ker(B_k - I)$, where

$$[B_k] - 2I = \begin{bmatrix} 0 & 0 & 0 \\ k & -1 & 0 \\ 5 & k-2 & -1 \end{bmatrix} \quad \text{and} \quad [B_k] - I = \begin{bmatrix} 1 & 0 & 0 \\ k & 0 & 0 \\ 5 & k-2 & 0 \end{bmatrix}.$$

So, we have to compute the dimensions of V_2 and V_1. A vector with coordinates (a, b, c) belongs to V_2 if and only if

$$\begin{bmatrix} 0 & 0 & 0 \\ k & -1 & 0 \\ 5 & k-2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

that is to say $ka - b = 0$ and $5a + (k-2)b - c = 0$. The value of a can be chosen freely, but after that choice the values of b and c are uniquely determined, since we have to put $b = ka$ and $c = (k^2 - 2k + 5)a$. It follows that $\dim V_2 = 1$ (i.e. the number of free choices), independently from k.

For V_1 the computation is similar. A vector (a, b, c) belongs to V_1 if and only if $a = 0$, $ka = 0$ and $5a + (k-2)b = 0$. The variable c does not appear in these equations, so its value can be chosen freely. On the other hand a must be 0, so that $(k-2)b = 0$. We have to analyze two cases:

- if $k = 2$, we can choose freely the value of b, so $\dim V_1 = 2$;
- if $k \neq 2$, we must set $b = 0$ and the only free choice is that of c, hence $\dim V_1 = 1$.

Comparing the dimensions of V_2 and V_1 with the algebraic multiplicities $m_2 = 1$ and $m_1 = 2$, we find that

- if $k = 2$ the algebraic multiplicities and the geometric ones (i.e. the dimensions of the eigenspaces) are the same and B_k is diagonalizable;
- if $k \neq 2$ we have $m_1 \neq \dim V_1$, so B_k is *not* diagonalizable.

(b) It is false. To prove this, let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an endomorphism such that the matrix associated to it with respect to some basis $\mathcal{B}$ is

$$[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

We will show that $L \neq B_3$ in any case. In fact, let $v_1, v_2 \in \mathbb{R}^3$ be the vectors that have coordinates respectively $(1, 0, 0)_{\mathcal{B}}$ and $(0, 1, 0)_{\mathcal{B}}$ with respect to the basis $\mathcal{B}$, and notice that L sends them to themselves, i.e. $Lv_1 = v_1$ and $Lv_2 = v_2$. (It is easy to deduce this: look at the first two columns of $[L]_{\mathcal{B}}$...) This implies that the dimension of the eigenspace of L relative to the eigenvalue 1, that is to say, the dimension of $V_1 = \{v \in \mathbb{R}^3 \mid Lv = v\}$, is at least two, because it contains the two vectors v_1 and v_2 that are linearly independent. From the previous question, we know that for $k \neq 2$, and in particular for $k = 3$, the dimension of the eigenspace of B_3 relative to the eigenvalue 1 is one, thus $L \neq B_3$.

Alternatively, notice that L is diagonalizable, whereas B_3 is not. ■

Exercise 5.13. Let $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the endomorphism whose associated matrix (with respect to the standard basis) is

$$[S] = \begin{bmatrix} 2 & a & 1 \\ 1 & a+1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

where a is a real parameter.

(a) For which values of a is S diagonalizable?

(b) For $a = -1$, find an orthonormal basis of the eigenspace $V_1 = \{v \in \mathbb{R}^3 \mid S(v) = v\}$ (with respect to the standard inner product on $\mathbb{R}^3$).

Exercise 5.14. Let $a \in \mathbb{R}$ be a real parameter ad let L_a be the linear endomorphism of $\mathbb{R}^3$ defined as

$$L_a\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2ax + y + z \\ x + ay + z \\ -x + y + az \end{bmatrix}.$$

Study the diagonalizability of L_a, depending on the parameter a.

Solution. The matrix associated to L_a with respect to the standard basis of $\mathbb{R}^3$ is

$$[L_a] = \begin{bmatrix} 2a & 1 & 1 \\ 1 & a & 1 \\ -1 & 1 & a \end{bmatrix}$$

and the characteristic polynomial is

$$p_a(\lambda) = \det \begin{bmatrix} 2a - \lambda & 1 & 1 \\ 1 & a - \lambda & 1 \\ -1 & 1 & a - \lambda \end{bmatrix}.$$

If we use the Laplace expansion on the first row we obtain

$$\begin{aligned} p_a(\lambda) &= (2a - \lambda)\big((a - \lambda)^2 - 1\big) - (a - \lambda + 1) + (1 + a - \lambda) = \\ &= (2a - \lambda)(a - \lambda + 1)(a + \lambda + 1) \end{aligned}$$

whose roots are all real and given by

$$\lambda_1 = 2a, \quad \lambda_2 = a - 1, \quad \lambda_3 = a + 1.$$

These are the eigenvalues of L_a; we study what happens if some of these values are equal.

First of all, notice that $\lambda_2 \neq \lambda_3$ for every a. If $a = 1$, then $\lambda_1 = \lambda_3 = 2$ which is an eigenvalue with algebraic multiplicity 2; if $a = -1$, then $\lambda_1 = \lambda_2 = -2$ which is an eigenvalue with algebraic multiplicity 2. So we have to distinguish several cases.

- If $a \neq \pm 1$, L_a is diagonalizable since it has three different eigenvalues, each with algebraic multiplicity 1.
- If $a = 1$, we have to compute the geometric multiplicity of the eigenvalue 2, i.e. $\dim\ker(L_1 - 2I)$. The associated matrix is

$$[L_1] - 2I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

  which has rank 2, so $\dim\ker(L_1 - 2I) = 1$ which is different to the algebraic multiplicity of the eigenvalue 2. It follows that L_1 is *not* diagonalizable.
- The case $a = -1$ is similar to the previous one: we have to compute the geometric multiplicity of the eigenvalue -2, i.e. $\dim\ker(L_{-1} + 2I)$. We get

$$[L_{-1}] + 2I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

  and this matrix also has rank 2. Thus the algebraic and geometric multiplicities of the eigenvalue -2 are different and the endomorphism L_{-1} is *not* diagonalizable. ■

Exercise 5.15. Let $a \in \mathbb{R}$ be a real parameter ad let F_a be the linear endomorphism of $\mathbb{R}^3$ that has as associated matrix (with respect to the standard basis)

$$\begin{bmatrix} a & 0 & 0 \\ 2 & 1 & -a \\ 3 & -a & 1 \end{bmatrix}.$$

Study the diagonalizability of F_a, depending on the parameter a.

Solution. The characteristic polynomial of F_a is

$$p_a(\lambda) = \det \begin{bmatrix} a - \lambda & 0 & 0 \\ 2 & 1 - \lambda & -a \\ 3 & -a & 1 - \lambda \end{bmatrix}$$

which reduces to (use Laplace expansion on the first row)

$$p_a(\lambda) = (a-\lambda)\big((1-\lambda)^2 - a^2\big) = -(\lambda - a)\big(\lambda - (1+a)\big)\big(\lambda - (1-a)\big).$$

The three eigenvalues are $\lambda_1 = a$, $\lambda_2 = 1 + a$, $\lambda_3 = 1 - a$. We study when these values are different.

- If $a = 1/2$, we have $\lambda_1 = \lambda_3 = 1/2$ and $\lambda_2 = 3/2$. In order to study diagonalizability, we have to check if for each eigenvalue λ the algebraic multiplicity m_λ and the geometric one g_λ are the same. Since for any eigenvalue we have $1 \le g_\lambda \le m_\lambda$, the only problem arises if $g_{1/2} = 1$ (in fact, $1 \le g_{3/2} \le m_{3/2} = 1$, whereas $m_{1/2} = 2$). The matrix associated to $F_{1/2} - (1/2)I$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1/2 & -1/2 \\ 3 & -1/2 & 1/2 \end{bmatrix}$$

  and its rank is 2, so $g_{1/2} = \dim\ker(F_{1/2} - (1/2)I) = 3 - 2 = 1$; it follows that $F_{1/2}$ is not diagonalizable.

- If $a = 0$, we have $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 1$. In this case the eigenvalue 0 has $m_0 = 1$ and the eigenvalue 1 has $m_1 = 2$, thus we can proceed in a similar way to the previous case and study the geometric multiplicity g_1:

$$[F_0 - I] = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix},$$

  therefore $\dim\ker(F_0 - I) = 2$ and $g_1 = m_1$. It follows that F_0 is diagonalizable.

- If $a \neq 0$ and $a \neq 1/2$, the three eigenvalues λ_1, λ_2 and λ_3 are distinct, so F_a is diagonalizable. ■

Exercise 5.16. Let $a \in \mathbb{R}$ be a real parameter ad let F_a be the linear endomorphism of $\mathbb{R}^3$ that has as associated matrix (with respect to the standard basis)

$$\begin{bmatrix} a & 0 & 0 \\ -2 & a & -2 \\ 0 & 2 & 4 \end{bmatrix}.$$

Study the diagonalizability of F_a, depending on the parameter a.

Exercise 5.17. Let $a \in \mathbb{R}$ be a real parameter ad let T_a be the linear endomorphism of $\mathbb{R}^3$ that has as associated matrix (with respect to the standard basis)

$$\begin{bmatrix} 0 & -3a & 0 \\ 1 & 2a & 0 \\ 1 & -3 & 1 \end{bmatrix}.$$

Study the diagonalizability of T_a over $\mathbb{R}$, depending on the parameter a.

Solution. The characteristic polynomial of T_a is

$$p_a(\lambda) = \det \begin{bmatrix} -\lambda & -3a & 0 \\ 1 & 2a-\lambda & 0 \\ 1 & -3 & 1-\lambda \end{bmatrix} = (1-\lambda)(\lambda^2 - 2a\lambda + 3a)$$

obtained by using Laplace expansion on the third column. The (reduced) discriminant of $\lambda^2 - 2a\lambda + 3a$ is $\Delta = a^2 - 3a = a(a-3)$, therefore the eigenvalues of T_a are all real if and only if $a \leq 0$ or $a \geq 3$, and in this case we have $\lambda_1 = 1$, $\lambda_2 = a + \sqrt{\Delta}$ and $\lambda_3 = a - \sqrt{\Delta}$.

If $a = 0$ or $a = 3$ we have $\lambda_2 = \lambda_3 = a$; we study for which other values of a we have an eigenvalue with algebraic multiplicity greater than one.

- If $a < 0$, we test whether $\lambda_1 = \lambda_2$ (notice that $\lambda_1 \neq \lambda_3$ in this case, since $a < 0$ implies $\lambda_3 < 0$). We get

$$1 = a + \sqrt{a^2 - 3a} \quad \Rightarrow \quad 1 - a = \sqrt{a^2 - 3a}$$

  and, since both sides are positive, we can square:

$$(1-a)^2 = a^2 - 3a$$

  that is true for $a = -1$.

- If $a > 3$, we have $\lambda_2 > 3$, so we only have to check if $\lambda_1 = \lambda_3$ can happen. We get

$$1 = a - \sqrt{a^2 - 3a} \quad \Rightarrow \quad \sqrt{a^2 - 3a} = a - 1$$

  and we can square as above:

$$a^2 - 3a = (a-1)^2$$

  that is true for $a = -1$, but this can't be because we are studying the case $a > 3$.

We reach a first conclusion: if $0 < a < 3$, T_a is not diagonalizable because not all eigenvalues are real; if $a < 0$ and $a \neq -1$, or $a > 3$, T_a is diagonalizable because the eigenvalues are real and distinct. The only remaining cases are the ones with a double eigenvalue, i.e. $a = -1$, $a = 0$ and $a = 3$.

- If $a = -1$, the eigenvalue 1 has algebraic multiplicity $m_1 = 2$. We compute the geometric one:

$$g_1 = \dim\ker(T_{-1} - I) = \dim\ker \begin{bmatrix} -1 & 3 & 0 \\ 1 & -3 & 0 \\ 1 & -3 & 0 \end{bmatrix} = 2$$

  so T_{-1} is diagonalizable.

- If $a = 0$, the eigenvalue 0 has algebraic multiplicity $m_0 = 2$. We compute the geometric one:

$$g_0 = \dim\ker(T_0) = \dim\ker \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -3 & 1 \end{bmatrix} = 1$$

  so $m_0 > g_0$ and T_0 is not diagonalizable.

- If $a = 3$, the eigenvalue 3 has algebraic multiplicity $m_3 = 2$. We compute the geometric one:

$$g_3 = \dim\ker(T_3 - 3I) = \dim\ker \begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & -3 & 2 \end{bmatrix} = 1$$

  so $m_3 > g_3$ and T_3 is not diagonalizable. ■

Exercise 5.18. Let $a \in \mathbb{R}$ be a real parameter ad let T_a be the linear endomorphism of $\mathbb{R}^3$ that has as associated matrix (with respect to the standard basis)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ a^2 & 0 & 1 \end{bmatrix}.$$

Study the diagonalizability of T_a over $\mathbb{R}$, depending on the parameter a.

Solution. The characteristic polynomial of T_a is

$$p_a(\lambda) = \det([T_a] - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & a-\lambda & 0 \\ a^2 & 0 & 1-\lambda \end{bmatrix} =$$
$$= -(\lambda - a)(\lambda - 1 + a)(\lambda - a - 1),$$

therefore the eigenvalues of T_a are $\lambda_1 = a$, $\lambda_2 = 1 - a$ and $\lambda_3 = 1 + a$. We have to study when these values coincide. Notice that $\lambda_1 \neq \lambda_3$ for all a, so only the following cases are possible:

- $\lambda_1 = \lambda_2$ if and only if $a = 1 - a$, that is $a = 1/2$;
- $\lambda_2 = \lambda_3$ if and only if $1 - a = 1 + a$, that is $a = 0$.

So, if $a \neq 0$ and $a \neq 1/2$, T_a has three distinct real eigenvalues and it is diagonalizable. Let us consider now the other cases.

- If $a = 0$, the eigenvalue 1 has algebraic multiplicity $m_1 = 2$. We compute its geometric multiplicity:

$$g_1 = \dim\ker([T_0] - I) = \dim\ker \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1$$

  because the rank of $[T_0] - I$ is 2 (the second and third columns are linearly independent). Therefore $g_1 \neq m_1$ and T_0 is not diagonalizable.

- If $a = 1/2$, the eigenvalue $1/2$ has algebraic multiplicity $m_{1/2} = 2$. We compute its geometric multiplicity:

$$g_{1/2} = \dim\ker\left([T_{1/2}] - \frac{1}{2}I\right) = \dim\ker\begin{bmatrix}1/2 & 0 & 1\\ 0 & 0 & 0\\ 1/4 & 0 & 1/2\end{bmatrix} =$$
$$= \dim\ker\begin{bmatrix}1/2 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = 2.$$

In this case $m_{1/2} = g_{1/2} = 2$ and $T_{1/2}$ is diagonalizable. ■

Exercise 5.19. Let $T_a \colon \mathbb{R}^3 \to \mathbb{R}^3$ an endomorphism represented by the matrix

$$[T_a] = \begin{bmatrix}0 & 2a & a\\ 0 & a+2 & 0\\ a & -2 & a^2-1\end{bmatrix}$$

with respect to the standard basis.

(a) Determine the values of $a \in \mathbb{R}$ for which T_a is diagonalizable.

(b) Determine the values of $a \in \mathbb{R}$ for which T_a is invertible.

Exercise 5.20. Let $A \in \mathcal{M}_4(\mathbb{R})$ be the matrix

$$A = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -a^2 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & a^2+1 & 0\end{bmatrix}.$$

Study the diagonalizability of A over the field $\mathbb{R}$ depending on the parameter $a \in \mathbb{R}$.

Solution. The characteristic polynomial of A is

$$p_A(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix}-\lambda & 1 & 0 & 0\\ 0 & -\lambda & -a^2 & 0\\ 0 & 0 & -\lambda & 1\\ 1 & 0 & a^2+1 & -\lambda\end{bmatrix} =$$
$$= -\lambda\det\begin{bmatrix}-\lambda & -a^2 & 0\\ 0 & -\lambda & 1\\ 0 & a^2+1 & -\lambda\end{bmatrix} - 1\det\begin{bmatrix}0 & -a^2 & 0\\ 0 & -\lambda & 1\\ 1 & a^2+1 & -\lambda\end{bmatrix} =$$
$$= -\lambda(-\lambda^3 - (-\lambda)(a^2+1)) - (-a^2) =$$
$$= \lambda^4 - (a^2+1)\lambda^2 + a^2$$

where we used the Laplace expansion on the first row (other choices are possible). In order to find its roots we put $\lambda^2 = t$ and we get the quadratic equation

$$t^2 - (a^2+1)t + a^2 = 0,$$

whose solutions are $t_1 = 1$ and $t_2 = a^2$. Therefore the solutions of $p_A(\lambda) = 0$ are given by $\lambda = \pm\sqrt{t_1}$ and $\lambda = \pm\sqrt{t_2}$, i.e.

$$\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = a,\ \lambda_4 = -a.$$

These solutions are always distinct, unless $a \in \{0, 1, -1\}$. Thus we may conclude that for $a \notin \{0, 1, -1\}$ A is diagonalizable, and study the other cases separately.

- If $a = 0$, the eigenvalues are 1, -1 and 0 with algebraic multiplicities $m_1 = m_{-1} = 1$ and $m_0 = 2$. Since the geometric multiplicity is always less than or equal to the algebraic one, the only computation we have to do is that of g_0, the geometric multiplicity of the eigenvalue 0.

  $$\dim\ker(A - 0I) = \dim\ker\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 1 & 0\end{bmatrix} \overset{(\star)}{=} \dim\ker\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} = 1$$

  where we performed the usual row-reduction algorithm in $(\star)$. Therefore $g_0 \neq m_0$ and A is not diagonalizable.

- Notice that the matrix A is the same both for $a = 1$ and for $a = -1$, so we can study these two cases together. The eigenvalues are 1 and -1, each with algebraic multiplicity $m_{\pm 1} = 2$, and we have to compute the two geometric multiplicities. Let us begin with g_1:

  $$\dim\ker(A - I) = \dim\ker\begin{bmatrix}-1 & 1 & 0 & 0\\ 0 & -1 & -1 & 0\\ 0 & 0 & -1 & 1\\ 1 & 0 & 2 & -1\end{bmatrix} = 1.$$

  From this computation we see that $g_1 \neq m_1$, so we can immediately conclude that A is not diagonalizable. ■

Exercise 5.21. **(I)** Let $A\colon \mathbb{C}^2 \to \mathbb{C}^2$ the endomorphism associated to the matrix

$$\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}$$

with respect to the standard basis. Find the eigenvectors of A and determine if A is diagonalizable.
(II) Consider the endomorphism $B\colon \mathbb{R}^2 \to \mathbb{R}^2$ associated to the matrix

$$\begin{bmatrix}1 & b\\ b & 1\end{bmatrix}$$

with respect to the standard basis, where $b \in \mathbb{R}$ is a parameter.

(a) For which values of $b \in \mathbb{R}$ is B diagonalizable?

(b) Let k be a positive integer. Determine, as a function of k and b, the eigenvalues of the endomorphism B^k.

Solution. **(I)** A brief computation gives us the characteristic polynomial

$$p_A(\lambda) = \lambda^2 - 5\lambda + 2,$$

whose roots are $1 + 2i$ and $1 - 2i$. Hence the endomorphism A is diagonalizable, because its eigenvalues are distinct. The eigenvectors relative to the eigenvalue $1 + 2i$ are the non-zero vectors in

$$\ker(A - (1 + 2i)I) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2 \;\middle|\; \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

It is not difficult to see that this eigenspace has dimension 1 and it is generated, for example, by the vector $(i, 1)$.

In a similar way, the eigenvectors relative to the eigenvalue $1 - 2i$ are the non-zero vectors in $\ker(A - (1 - 2i)I)$, and it turns out to be a 1-dimensional subspace generated by the vector $(i, -1)$.

(II) (a) The characteristic polynomial of B is $p_B(\lambda) = \lambda^2 - 2\lambda + 1 - b^2$, whose roots are $1+b$ and $1-b$. It follows immediately that, if $b \neq 0$, the two eigenvalues are different, so B is diagonalizable. Otherwise, if $b = 0$, to answer the question we could compute the geometric multiplicity of the only eigenvalue 1. However, notice that for $b = 0$ the matrix associated to B is the identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is obviously diagonalizable—in fact, it is already diagonal! In conclusion, the endomorphism B is diagonalizable for every value of b.

(b) In order to compute B^k, we choose a basis $\mathcal{B}$ such that the associated matrix of B with respect to $\mathcal{B}$ is diagonal:

$$[B]_{\mathcal{B}} = \begin{bmatrix} 1+b & 0 \\ 0 & 1-b \end{bmatrix}$$

(recall that in this case the diagonal elements are exactly the eigenvalues of B). By induction on k we have that

$$([B]_{\mathcal{B}})^k = [B^k]_{\mathcal{B}} = \begin{bmatrix} (1+b)^k & 0 \\ 0 & (1-b)^k \end{bmatrix},$$

so the eigenvalues of B^k are $(1 + b)^k$ and $(1 - b)^k$. ■

Exercise 5.22. Let B be the matrix

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

(a) Prove that $\lambda_1 = 1$ is an eigenvalue of B with geometric multiplicity $g_1 = 3$. Use this fact to deduce that the characteristic polynomial of B factors as a product of polynomials of degree 1.

(b) Say if there exists an eigenvalue of B different from 1. [**Hint:** you don't need to compute the characteristic polynomial of B—there are faster methods.] Is B diagonalizable?

Solution. (a) To answer to the first part of the question, it is sufficient to verify that $\dim \ker(B - I) = 3$. Now,

$$B - I = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

which can be reduced in echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is only one pivot, so $\operatorname{rk}(B - I) = 1$ and $\dim \ker(B - I) = 4 - 1 = 3$.

From the previous result we deduce that the algebraic multiplicity of the eigenvalue 1 is at least 3, therefore there are only two possibilities for the characteristic polynomial: either $p_B(\lambda) = (\lambda - 1)^3(\lambda - \lambda_2)$ for some λ_2, or $p_B(\lambda) = (\lambda - 1)^4$. In both cases, p_B has only linear factors. Notice that $\lambda_2 \in \mathbb{R}$, because for a matrix with real coefficients the complex eigenvalues are always pairwise conjugated, if they exist.

(b) The trace of B is $\operatorname{tr} B = 8$, and it is the sum of the eigenvalues (counted with multiplicity). Thus we can find the last eigenvalue λ_2 by computing

$$8 = 3 \cdot 1 + \lambda_2 \quad \Rightarrow \quad \lambda_2 = 5.$$

A quick computation gives $\dim \ker(B - 5I) = 1$. In fact,

$$\ker(B - 5I) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right).$$

In conclusion, since $m_1 = g_1 = 3$ and $m_5 = g_5 = 1$, B is diagonalizable.

Otherwise, we can say immediately that B is diagonalizable because it is symmetric. ■

Chapter 6

Scalar Products

Exercise 6.1. Let v_1, v_2 be the following vectors of $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}.$$

(a) Find an orthogonal basis (q_1, q_2) of $\mathrm{span}(v_1, v_2)$.

(b) Find a vector $v \neq 0$ orthogonal to both v_1 and v_2.

Solution. We will denote with $\langle v, w \rangle$ the standard scalar product between v and w in $\mathbb{R}^3$.
(a) We use the Gram-Schmidt algorithm: set $q_1 = v_1$ and

$$\begin{aligned} q_2 = v_2 - \frac{\langle v_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 &= \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} - \frac{-1 \cdot 1 + 0 \cdot 1 - 2 \cdot 1}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \\ &= \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

(b) A generic vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is orthogonal to both v_1 and v_2, that is $\langle v, v_1 \rangle = 0$ and $\langle v, v_2 \rangle = 0$, if and only if

$$\begin{cases} x + y + z = 0 \\ -x - 2z = 0. \end{cases}$$

We solve the system with the usual method, obtaining the set of solutions

$$\left\{ \begin{bmatrix} -2z \\ z \\ z \end{bmatrix} \;\middle|\; z \in \mathbb{R} \right\} = \mathrm{span}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right).$$

Any non-zero vector in this set (e.g. $v = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$) answers the question.

Alternatively, we could proceed with Gram-Schmidt: take any vector $v_3 \notin \operatorname{span}(v_1, v_2)$ and set

$$v = v_3 - \frac{\langle v_3, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 - \frac{\langle v_3, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2. \qquad \blacksquare$$

Chapter 7

More Exercises

Exercise 7.1. **(I)** Determine the coordinates of the vector

$$v = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$$

with respect to the basis $\mathcal{B} = (v_1, v_2, v_3)$ of $\mathbb{R}^3$, where

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

(II) Let $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map whose associated matrix with respect to the basis $\mathcal{B}$ is

$$A = [f]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & -2 & 2 \\ 1 & -1 & 3 \end{bmatrix}.$$

Determine the coordinates of $f(v)$ with respect to the standard basis of $\mathbb{R}^3$.

Solution. **(I)** The coordinates of v are the coefficients of the linear combination

$$\begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

which are found by solving the system

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}.$$

We represent the system with the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ -1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 \end{array}\right].$$

By elementary *row* operations we don't change the set of solutions; after a bit of computation we get the row reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -8 \end{array}\right],$$

so the coordinates of v with respect to $\mathcal{B}$ are $(-11, 5, -8)$.
(II) Since A represents f with respect to the basis $\mathcal{B}$ and now we know the coordinates of v with respect to the same basis, it is easy to compute the coordinates of $f(v)$ with respect to $\mathcal{B}$:

$$[f(v)]_{\mathcal{B}} = [f]_{\mathcal{B}}[v]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & -2 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -11 \\ 5 \\ -8 \end{bmatrix} = \begin{bmatrix} -27 \\ -15 \\ -40 \end{bmatrix}.$$

This means that $f(v) = -27v_1 - 15v_2 - 40v_3$. We already know the coordinates of v_1, v_2, v_3 with respect to the standard basis, so we can conclude

$$f(v) = -27 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 15 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 40 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -13 \\ -55 \end{bmatrix}. \qquad \blacksquare$$

Exercise 7.2. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z + y \\ y + x \\ 2x \end{bmatrix}.$$

(a) Compute the matrix $A = [f]$ associated to f with respect to the standard basis of $\mathbb{R}^3$.

(b) Compute the matrix $M = [f]_{\mathcal{B}}^{\mathcal{B}}$ associated to f with respect to the basis $\mathcal{B} = (v_1, v_2, v_3)$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

(c) Compute *one* eigenvalue of f.

Solution. In general, recall that if $g\colon V \to W$ is a linear map and $\mathcal{V}, \mathcal{W}$ are bases respectively of V and W, the matrix $[g]_{\mathcal{V}}^{\mathcal{W}}$ associated to g has in the i-th column the coordinates of $g(v_i)$ with respect to the basis $\mathcal{W}$, where v_i is the i-th vector of $\mathcal{V}$.
(a) If $\mathcal{C} = (e_1, e_2, e_3)$ is the standard basis, we compute

- $f(e_1) = f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\2\end{bmatrix}$;
- $f(e_2) = f\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\end{bmatrix}$;
- $f(e_3) = f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$;

therefore

$$A = \begin{bmatrix}0 & 1 & 1\\1 & 1 & 0\\2 & 0 & 0\end{bmatrix}.$$

(b) We have to compute $f(v_i)$ for $i = 1, 2, 3$.

- $f(v_1) = f\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\2\\2\end{bmatrix}$ and we easily see that $f(v_1) = 2v_1 + 0v_2 + 0v_3$, so the first column of M is $\begin{bmatrix}2\\0\\0\end{bmatrix}$.
- $f(v_2) = f\left(\begin{bmatrix}2\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\4\end{bmatrix}$ and the coordinates with respect to $\mathcal{B}$ are not computed as easily as the previous point. We have to find $a, b, c \in \mathbb{R}$ such that $av_1 + bv_2 + cv_3 = f(v_2)$, i.e. we have to solve the system

$$\begin{bmatrix}1 & 2 & -1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}1\\3\\4\end{bmatrix}.$$

  The computation is not difficult and it turns out that $f(v_2) = 4v_1 - v_2 + v_3$.
- $f(v_3) = f(-e_1) = -f(e_1) = \begin{bmatrix}0\\-1\\-2\end{bmatrix}$. With a computation like the one above, we obtain $f(v_3) = -2v_1 + v_2 + 0v_3$.

Therefore

$$M = \begin{bmatrix}2 & 4 & -2\\0 & -1 & 1\\0 & 1 & 0\end{bmatrix}.$$

Otherwise, we may use the formula

$$[f]_{\mathcal{B}}^{\mathcal{B}} = [\mathrm{id}]_{\mathcal{C}}^{\mathcal{B}}[f][\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}}$$

with

$$[\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}} = \left[\begin{array}{c|c|c} v_1 & v_2 & v_3 \end{array}\right] = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and $[\mathrm{id}]_{\mathcal{C}}^{\mathcal{B}} = ([\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}})^{-1}$, which we find with the usual method:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

swap (third row) and (first row)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & -1 & 1 & 0 & 0 \end{array}\right]$$

−(third row)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -2 & 1 & -1 & 0 & 0 \end{array}\right]$$

(second row) − (first row)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ -1 & -2 & 1 & -1 & 0 & 0 \end{array}\right]$$

(third row) + (first row)
(third row) + 2(second row)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

(c) In the computation of M, we found that $f(v_1) = 2v_1$. Therefore 2 is an eigenvalue, with eigenvector v_1. ■

Exercise 7.3. Let $V \subseteq \mathbb{R}^4$ be the subspace $V = \operatorname{span}(v_1, v_2)$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Verify that (v_1, v_2) is a basis of V and extend it to a basis of $\mathbb{R}^4$.

(b) Does it exist a linear map $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ such that $\ker T = \operatorname{Im} T = V$? If not, explain why, otherwise choose a basis $\mathcal{B}$ of $\mathbb{R}^4$ and write the matrix $[T]^{\mathcal{B}}_{\mathcal{B}}$ associated to T with respect to this basis.

Solution. (a) We only need to check if v_1 and v_2 are linearly independent. We row-reduce the matrix whose columns are v_1 and v_2:

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The pivots are in both columns, so v_1 and v_2 are indeed linearly independent. In order to extend (v_1, v_2) to a basis of $\mathbb{R}^4$, we row-reduce the matrix

$$\left[\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

After a quick computation, the row reduced form is

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which has pivots in the 1st, 2nd, 3rd and 6th columns. Therefore the set

$$\mathcal{B} = \left(v_1, v_2, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

is a basis of $\mathbb{R}^4$ that extends (v_1, v_2).
(b) We choose the basis $\mathcal{B}$ of point (a). Since we must have $\ker T = V$, we define $T(v_1) = 0$ and $T(v_2) = 0$. On the other hand

$$\operatorname{Im} T = \operatorname{span}(T(v_1), T(v_2), T(v_3), T(v_4)) = \operatorname{span}(T(v_3), T(v_4))$$

and this has to be equal to $V = \operatorname{span}(v_1, v_2)$. The easiest way is to define $T(v_3) = v_1$ and $T(v_4) = v_2$. The associated matrix with respect to $\mathcal{B}$ is

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We know for sure that $\operatorname{Im} T = \operatorname{span}(v_1, v_2) = V$ and $\ker T \supseteq \operatorname{span}(v_1, v_2) = V$. But $\dim \ker T + \dim \operatorname{Im} T = 4$, so $\ker T$ can't contain V properly. ■

Exercise 7.4. Let C be the matrix

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Find a basis of $U = \ker C$ and a basis of $W = \operatorname{Im} C$.

(b) Find a set of cartesian equations for W, i.e. write W as the set of solutions of a homogeneous linear system.

Solution. (a) We row-reduce C in echelon form:

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivots are in the first and second columns of $\tilde{C}$, so the respective columns of C form a basis of $\operatorname{Im} C$:

$$W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right).$$

To find a basis of $\ker C$, we solve the system associated to $\tilde{C}$:

$$\begin{cases} x + z = 0 \\ y + t = 0 \end{cases}$$

from which we get

$$U = \ker C = \left\{ \begin{bmatrix} -z \\ -t \\ z \\ t \end{bmatrix} \;\middle|\; z, t \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

(b) A generic vector with coordinates (x, y, z) belongs to W if and only if the matrix

$$\begin{bmatrix} 1 & 1 & x \\ 0 & 1 & y \\ 1 & 0 & z \end{bmatrix}$$

has *not* maximum rank. Its row reduced echelon form is

$$\begin{bmatrix} 1 & 1 & x \\ 0 & 1 & y \\ 0 & 0 & -x+y+z \end{bmatrix}$$

which has not rank 3 if and only if $-x + y + z = 0$. Therefore

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; -x+y+z=0 \right\}.$$ ■

Exercise 7.5. Consider the following two subspaces of $\mathbb{R}^3$:

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x+y+z=0 \right\} \quad \text{and} \quad W = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right).$$

(a) Compute a basis of $V + W$ and say if they are in direct sum.

(b) Find a linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\ker f = V$ and $\operatorname{Im} f = W$.

(c) Prove that the set

$$\{f\colon \mathbb{R}^3 \to \mathbb{R}^3 \mid f \text{ is linear, } V \subseteq \ker f \text{ and } \operatorname{Im} f \subseteq W\}$$

is a linear subspace of the space of all linear endomorphisms of $\mathbb{R}^3$.

Exercise 7.6. Let V be the subspace of $\mathbb{R}[x]_{\leq 5}$ (polynomials with real coefficients and degree less than or equal to 5) defined as

$$V = \{p(x) \in \mathbb{R}[x]_{\leq 5} \mid p(1) = p(-1) = 0\}.$$

(a) Determine a basis for V over $\mathbb{R}$.

(b) Determine whether there exist injective or surjective linear maps from V to the space W of 2×2 symmetric matrices with real coefficients, and explain why.

Solution. A generic 2×2 symmetric matrix with real coefficients has the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

with $a, b, c \in \mathbb{R}$, so the space W has dimension 3 over $\mathbb{R}$ and a basis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In order to study the existence of injective or surjective linear maps from V to W, we have to compare their dimensions. In particular,

- there exist *injective* maps from V to W if and only if $\dim V \leq \dim W$ (send a basis of V injectively to a set of linearly independent vectors of W);
- there exist *surjective* maps from V to W if and only if $\dim V \geq \dim W$ (send a basis of V surjectively to a basis of W).

We have to compute $\dim V$, and to do so we will find a basis of V, thus answering (a) and (b) at the same time.

Recall that α is a root of a polynomial p (i.e. $p(\alpha) = 0$) if and only if $(x - \alpha)$ is a factor of $p(x)$, that is to say that there exists a polynomial $q(x)$ such that $p(x) = (x - \alpha)q(x)$. Therefore the condition $p(1) = p(-1) = 0$ is equivalent to the fact that $(x - 1)$ and $(x + 1)$ are factors of $p(x)$. Then, for any $p(x) \in V$ we can find a polynomial $q(x)$ such that

$$p(x) = (x - 1)(x + 1)q(x)$$

and, since $\deg p = 2 + \deg q$, we have also $\deg q \leq 3$. It is easy to see that this map from V to the space $\mathbb{R}[x]_{\leq 3}$ is linear and bijective (given a polynomial $q(x)$ with $\deg q \leq 3$, its inverse image is the polynomial $(x - 1)(x + 1)q(x)$, that belongs to V), so it is an isomorphism of vector spaces. Now, $\dim(\mathbb{R}[x]_{\leq 3}) = 4$ and a basis is given by $\{x^3, x^2, x, 1\}$, therefore we can conclude that $\dim V = 4$ and a basis for V is

$$\{(x - 1)(x + 1)x^3, (x - 1)(x + 1)x^2, (x - 1)(x + 1)x, (x - 1)(x + 1)\}$$

obtained by taking the inverse images of the vectors in the basis of $\mathbb{R}[x]_{\leq 3}$.

Finally, since $\dim V$ is strictly greater than $\dim W$, we can say that there exist surjective linear maps from V to W but not injective ones. ■

Exercise 7.7. Let $V = \mathcal{M}_n(\mathbb{R})$ be the vector space of $n \times n$ matrices over the field $\mathbb{R}$. Let $\mathcal{S} = \{M \in V \mid M = M^{\mathrm{T}}\}$ be the subspace of symmetric matrices, and let $\mathcal{A} = \{M \in V \mid -M = M^{\mathrm{T}}\}$ be subspace of antisymmetric matrices.

(a) Prove that $V = \mathcal{S} \oplus \mathcal{A}$.

(b) Let $\varphi\colon V \to V$ the linear map defined as

$$\varphi(M) = M + M^{\mathrm{T}}.$$

Determine if φ is diagonalizable and, if so, compute a basis of V made of eigenvectors for φ.