RICEVIMENTO 07/04

$$A = \begin{bmatrix} -n & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & -n \end{bmatrix}$$

INPUT A, x

OUTPUT y

$$Ax = y$$

$$y_k = -n x_k + \sum_{\substack{i=1 \\ i \neq k}}^{n} x_i$$

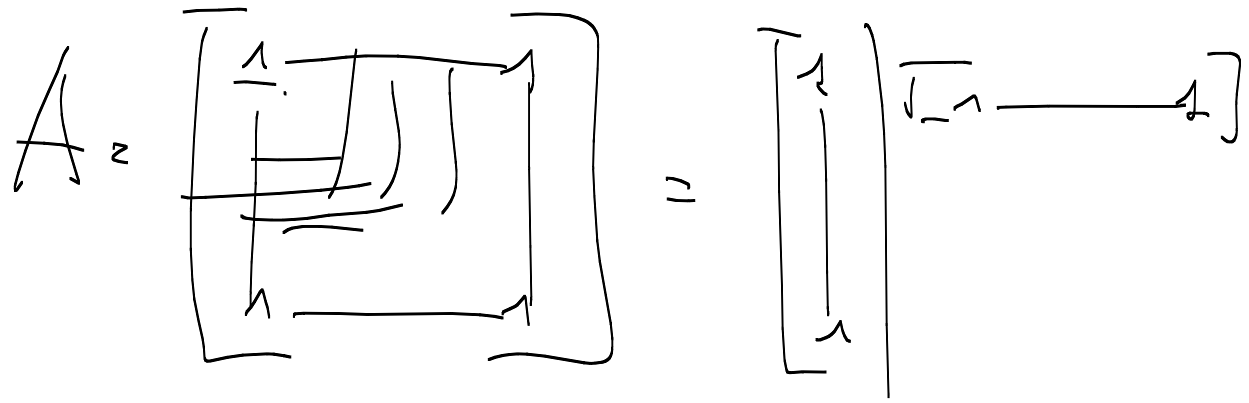
$$= \sum_{i=1}^{n} x_i - (n+1) x_k$$

$k = 1 \dots n$

① calcola $s = \sum_{i=1}^{n} x_i \Rightarrow O(n)$ ops

② for k=1:n; y(k)= s-(n+1)x(k);

end; $\overline{O(n) \text{ ops}}$



(1) A ammette fattorizzazione?

(2) In caso affermativo determinare la fattorizzazione.

Se $\det A_k \neq 0$ $k=1 \dots n-1$ $\Rightarrow$ $\exists!\, LU$

$$A \Rightarrow B$$

$$\sim B \Rightarrow \sim A$$

$$\cancel{\sim A \Rightarrow \sim B}$$

Provo a calcolare la fattorizzazione LU

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

$a_{11}^{(0)} \neq 0 \Rightarrow$ posso fare il 1° passo.

$$E_1 = \begin{bmatrix} 1 & & \\ -1 & \ddots & \\ \vdots & & \\ -1 & & 1 \end{bmatrix}$$

$$A^{(1)} = E_1 \cdot A^{(0)} = E_1 A$$

$$= \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$$ triangolare superiore.

e quindi mi posso fermare dopo il 1° passo

$$E_1 A = U \iff A = E_1^{-1} U$$

$$A = \begin{bmatrix} 1 & & \\ 1 & \ddots & \\ \vdots & & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

LU factorization.

$$A = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & & 0 \end{bmatrix}$$

also factorization LU

$$A = I + \alpha \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

$$A^{-1} = I + \beta \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

(1) Determina per quali valori di α A è singolare.

(2) Determina β t.c. $A^{-1} = I + \beta \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} (1 \cdots 1)$

$$A = \begin{bmatrix} \alpha+1 & \alpha & \cdots & \alpha \\ \alpha & \ddots & & \vdots \\ \vdots & & \ddots & \alpha \\ \alpha & \cdots & \alpha & \alpha+1 \end{bmatrix}$$

(1) $\underline{Ax = 0}$ $\iff$ $\begin{cases} x_1 + \alpha \sum_1^n x_i = 0 \\ \vdots \\ x_n + \alpha \sum_1^n x_i = 0 \end{cases}$

$\iff x_1 = x_2 = \cdots = x_n = x$

$\iff x + n\alpha x = 0 \iff$

$(1 + n\alpha) x = 0$

Se $1 + n\alpha \neq 0 \implies x = 0$ $\quad x_1 = x_2 = \cdots = x_n = 0$

Se $1 + n\alpha = 0 \iff \alpha = -\frac{1}{n}$ $\quad x_1 = x_2 = \cdots = x_n = 1$

A è singolare $\iff$ $\alpha = -\frac{1}{n}$

$$A = I + \alpha \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}$$

Chi sono gli autovalori di A $\qquad$ 1

$\dim \mathrm{Imm} + \dim \ker = n$

$$\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \quad \text{autovalori: } \begin{cases} 0 & \sigma = \tau = n-1 \\ n & \sigma = \tau = 1 \end{cases}$$

$$\alpha \cdot \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \quad \text{autovalori: } \begin{cases} 0 \\ \alpha n \end{cases}$$

$$A = I + \alpha \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \quad \text{autov. } 1 + n\alpha$$

$$1 + 0 = 1$$

$$1 + n\alpha = 0 \iff \alpha = -\frac{1}{n}$$

$$A = I + \alpha \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

$$A^{-1} = I + \beta \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

$$\left(I + \alpha e e^T\right) \cdot \left(I + \beta e e^T\right) = I$$

$$\cancel{I} + \beta e e^T + \alpha e e^T + \alpha\beta\, e(e^T e)e^T = \cancel{I}$$

$$\beta e e^T + \alpha e e^T + n\alpha\beta e e^T = 0$$

$$\underbrace{\left(\beta + n\alpha\beta + \alpha\right)}_{\text{scalare}} \underbrace{e e^T}_{\text{matrice}} = 0 \qquad \text{matrice} = \text{matrice nulla}$$

$$\iff \text{scalare} = 0$$

$$\beta(1 + n\alpha) + \alpha = 0$$

$$\beta(1 + n\alpha) = -\alpha$$

A e nesingulară $\Leftrightarrow 1 + n\alpha \neq 0$

$$\boxed{\beta = -\frac{\alpha}{1+n\alpha}}$$

$$A = \begin{bmatrix} \theta & & 1 \\ & \ddots & \\ 1 & & \theta \end{bmatrix}$$

(1) Fie θ a.î. să admită factorizare LU

Dacă $\underline{\theta \neq 0}$ $\det A_k \neq 0$ $k = 1, \dots, n-1$

$\Rightarrow$ Teorema de $\exists$ și unicitate a factorizării LU a lui A

Dacă $\theta = 0$ $\underline{\nexists\, LU}$

$$A = \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{bmatrix}$$ per assurdo

$$= L\,U = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} \bullet & & \\ & \ddots & \\ 0 & & \end{bmatrix} \Rightarrow a_{1,n} = 0$$

$\Rightarrow$ non $\exists$ fattorizzazione LU

$A \in \mathbb{R}^{n \times n}$ dire se $\exists$ fattorizzazione LU di A

$A = \begin{bmatrix} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$ trova la fattorizzazione LU

|| diagonalmente ~~dati i problemi di risoluzione di casi generici (,)~~

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad E_1 = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ -1 & & 1 \end{pmatrix}$$

$$E_1 A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Quindi data $A \in \mathbb{K}^{m \times n}$ si fattorizza con

$$A = \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$$

$$\underline{a}_{ij} = \begin{array}{c} j \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= 1$$

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = A^T$$

$$\text{Autovalori} \begin{cases} 0 & \sigma = \tau = n-1 \\ n & \sigma = \tau = 1 \end{cases}$$

$$Ax = \lambda x$$

$$Ax = 0 \iff \begin{cases} x_1 + x_2 + \dots + x_n = 0 \\ \vdots \\ x_1 + x_2 - - - + x_n = 0 \end{cases}$$

$$\iff x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ -x_1 - x_2 \cdots - x_{n-1} \end{bmatrix}$$

$$Ax = n x$$

$$\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = n \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

$$Ax = \lambda x \iff \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\iff (x_1 + x_2 + \cdots + x_n) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\underline{x_1 + x_2 + \cdots + x_n = 0} \qquad 0 = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow \lambda = 0$$

$$\lambda = 0 \qquad x_1 + \cdots + x_n = 0$$

$$x_1 + x_2 + \cdots + x_n \neq 0$$

$$\lambda = x_1 + x_2 + \cdots + x_n$$

$$\alpha \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \delta \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\delta \neq 0 \qquad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \frac{\alpha}{\delta} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = c \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$A\, \cancel{c} \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = n\, \cancel{c} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow \delta = n$$

$\delta = 0$ | autovett: $x_1 + x_2 + \dots + x_n = 0$

$\delta = n$ | autovett $c \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ $(c \neq 0)$

$A = LU$ suppongo che $\exists$ fattorizz. LU

$$\Rightarrow \quad A_k = L_k U_k \quad \Rightarrow$$

$$\det A_k = \det U_k = \prod_{j=1}^{k} u_{jj}$$

A_3 sia singolare $\Leftrightarrow$ $\det A_3 = 0$

$$u_{11} u_{22} u_{33} = 0$$

$\Leftrightarrow$ almeno 1 termine di prod $= 0$

A_5 non singolare

$$\det A_5 = \underline{u_{11} \ldots u_{55}} = 0$$