RICEVIMENTO 30/04

$$A = \begin{bmatrix} \alpha & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & \alpha \end{bmatrix}$$

(1) A simmetrica $\Rightarrow$ A ha autovalori reali

A è diagonalizzabile

$\Downarrow$

$\sigma_i = \gamma_i \quad \forall \lambda_i$ autovalore di A

λ autovalore di A

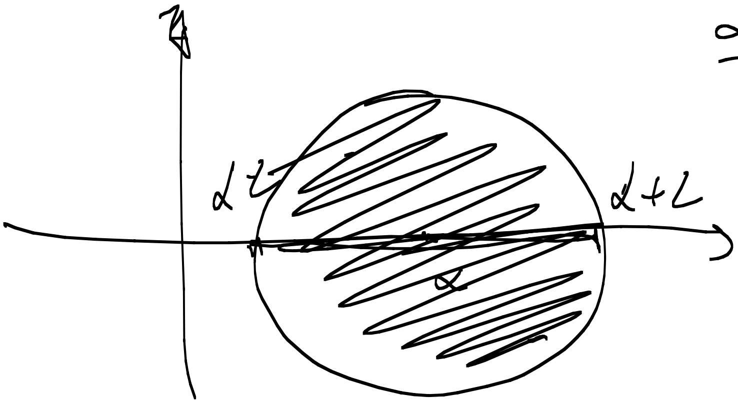
$$A - \lambda I = \begin{bmatrix} \alpha-\lambda & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & \alpha-\lambda \end{bmatrix}$$

$$\underbrace{\dim \operatorname{Ker}(A-\lambda I)}_{\gamma \le 1} + \underbrace{\dim \operatorname{Im}(A-\lambda I)}_{\ge n-1} = n$$

$\forall \lambda$ autovalore di A $\quad \gamma = \sigma = 1 \Rightarrow$ gli autovalori sono semplici

Ostrovski

$\alpha > 2$



$\alpha-2 \le \lambda_i \le \alpha+2$

$\alpha-2 \le \lambda_1 < \lambda_2 < \dots < \lambda_n \le \alpha+2$

$\|A\|_2 = \sqrt{\varphi(A^T A)} = \sqrt{\varphi(A^2)} = \sqrt{\lambda_n^2} = \lambda_n$

$\|A^{-1}\|_2 = \sqrt{\varphi(A^{-T}A^{-1})} = \sqrt{\varphi(A^{-2})} = \sqrt{\frac{1}{\lambda_1^2}} = \frac{1}{\lambda_1}$

$K_2(A) = \frac{\lambda_n}{\lambda_1} = \frac{\lambda_{max}}{\lambda_{min}}$

$\le \frac{\alpha+2}{\alpha-2}$

$$\|A^{-1}\| = \sqrt{\rho\left((A^{-1})^T A^{-1}\right)} = \sqrt{\rho\left(A^{-T} A^{-1}\right)}$$

$$A = A^T \Rightarrow A^{-1} = (A^{-T})^T$$

$$= \sqrt{\rho\left(A^{-1} \cdot A^{-1}\right)} = \sqrt{\rho\left(A^{-2}\right)}$$

$$A \cdot v = \lambda v \Rightarrow v = \lambda A^{-1} v$$

$$\Rightarrow A^{-1} v = \frac{1}{\lambda} v$$

$\lambda_1 < \lambda_2 < \dots < \lambda_n$ (autovalori di A)

$\frac{1}{\lambda_n} < \frac{1}{\lambda_{n-1}} \dots < \frac{1}{\lambda_2}$ (autovalori di A^{-1})

$\left(\frac{1}{\lambda_n}\right)^2 < \left(\frac{1}{\lambda_{n-1}}\right)^2 < \dots < \left(\frac{1}{\lambda_1}\right)^2$ (autovalori di A^{-2})

$$\|A^{-1}\| = \sqrt{\rho\left(A^{-2}\right)} = \sqrt{\left(\frac{1}{\lambda_1}\right)^2} = \frac{1}{\lambda_2}$$

(2) $\begin{bmatrix} \alpha & 1 & & \\ & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \alpha & \cdots & \alpha & \alpha \end{bmatrix} \in \mathbb{R}^{n \times n}$

(1) Per quali α A è p.d.

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad i=1 \cdots n$$

$i=1$ $|\alpha| > 1$

$i=2$ $|\alpha| > |\alpha| + 1$ Mai

Non esistono valori di α per cui A è p.d.

$n=2$ $\begin{bmatrix} \alpha & 1 \\ \alpha & \alpha \end{bmatrix}$ $|\alpha| > 1$, $|\alpha| > |\alpha|$ Mai

$n > 2$ $\begin{bmatrix} \alpha & 1 & 0 \\ \alpha & \alpha & 1 \\ \alpha & \alpha & \alpha \\ \vdots & & \vdots \end{bmatrix}$

Applicabilità (2) $a_{ii} \neq 0 \quad \forall i = 1 \dots n$

$\underline{\alpha \neq 0}$

GS $\quad M = \begin{bmatrix} \alpha & & \\ \vdots & \ddots & \\ \alpha & \cdots & \alpha \end{bmatrix} \quad N = \begin{bmatrix} & -1 & \\ & & \ddots \\ 0 & & & -1 \end{bmatrix}$

$G = M^{-1} N$

$M^{-1} = ?$ $\quad M = \alpha \cdot \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix}$

$$M^{-1} = \frac{1}{\alpha} \cdot \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix}^{-1} = \frac{1}{\alpha} \begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} = I$$

= prodotto triangolare inferiore

e diagonale principale = 1

$$i \neq j \qquad \begin{array}{c} \text{i-esima} \\ \downarrow \\ \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{c} j \\ \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ 0 \\ \vdots \end{bmatrix} \end{array} = 1 - 1 = 0$$

$$M^{-1} = \frac{1}{\alpha} \begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

$$G = M^{-1}N = \frac{1}{\alpha}\begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & \end{bmatrix}$$

$$= -\frac{1}{\alpha}\begin{bmatrix} 0 & -1 & 0 & \\ \vdots & 1 & -1 & \\ \vdots & & 1 & \ddots & -1 \\ 0 & & & & 1 \end{bmatrix}$$

G triangolare superiore

G ha autovalori 0, $\frac{1}{\alpha}$

$$\rho(G) = \frac{1}{|\alpha|}$$

G.S. è convergente $\iff \frac{1}{|\alpha|} < 1$

$\iff 1 < |\alpha| \iff |\alpha| > 1$

③ $\begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix}$

Jacobi $M_2 = \begin{bmatrix} 2 & & \\ & \ddots & \\ & & 2 \end{bmatrix}$ $N_2 = \begin{bmatrix} 0 & -1 & \cdots & -1 \\ & & 0 & \\ -1 & \cdots & -1 & 0 \end{bmatrix}$

$M^{-1} = \begin{bmatrix} \frac{1}{2} & & \\ & \ddots & \\ & & \frac{1}{2} \end{bmatrix}$

$-M^{-1}N_2 = \begin{bmatrix} 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ & & & \\ \frac{1}{2} & \cdots & \frac{1}{2} & 0 \end{bmatrix}$

$\|M^{-1}N\|_\infty = \frac{1}{2}(n-1)$

$\|M^{-1}N\|_1 = 1$

Chi sono gli autovalori di $M^{-1}N$

$$\det\left(\lambda I - M^{-1}N\right) = \det\begin{pmatrix} \lambda & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda & \\ \frac{1}{2} & \cdots & \cdots & \frac{1}{2} & \lambda \end{pmatrix}$$

$$= \lambda \det\begin{pmatrix} \lambda & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda & \\ \frac{1}{2} & \cdots & \cdots & \frac{1}{2} & \lambda \end{pmatrix} = \lambda^{n-2} \cdot \det\begin{bmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{bmatrix}$$

$$= \lambda^{n-2} \cdot \left(\lambda^2 - \frac{1}{4}\right) \quad \begin{cases} \lambda = 0 \\ \lambda = \pm\frac{1}{2} \end{cases}$$

$\rho(J) = \frac{1}{2} < 1$ il metodo è convergente.

$J^2 = ?$

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ & & & \\ \frac{1}{2} & \cdots & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ & & & \\ \frac{1}{2} & \cdots & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \cdots & \frac{1}{4} & 0 \\ & & & \\ 0 & \frac{1}{4} & \cdots & \frac{1}{4} \end{bmatrix}$$

$$\|J^2\|_1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1$$

$$e^{(k+1)} = J^{k+1} e^{(0)}$$

$$e^{(k)} = J^k \cdot e^{(0)}$$

$$e^{(2k)} = J^{2k} e^{(0)} = (J^2)^k e^{(0)}$$

$$\|e^{(2k)}\|_1 \leq \|J^2\|_1^k \|e^{(0)}\|_1$$

$$\|e^{(2k)}\|_1 \leq \left(\frac{1}{2}\right)^k \|e^{(0)}\|_1$$

$$\frac{\| e^{(2k)} \|_1}{\| e^{(0)} \|_1} \leq \left(\frac{1}{2}\right)^k$$

$$\left(\frac{1}{2}\right)^k \leq 2^{-40}$$

$$\frac{\| e^{(2k)} \|_1}{\| e^{(0)} \|_1} \leq \left(\frac{1}{2}\right)^k \leq 2^{-40}$$

$$\left(\frac{1}{2}\right)^k \leq 2^{-40} \iff 2^{-k} \leq 2^{-40}$$

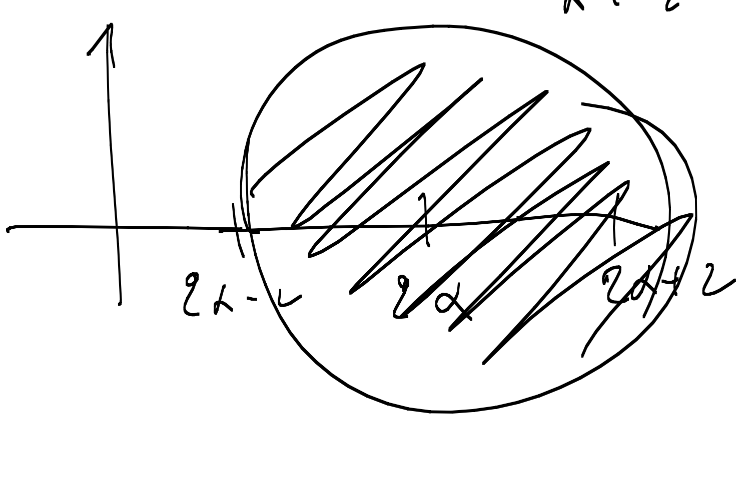
$$2^{40} \leq 2^k \iff k \geq 40$$

$$A = \begin{bmatrix} 2\alpha & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2\alpha \end{bmatrix}$$

Se $\alpha > 1$ allora A è def posit e Jacobi converge

A è def positiva se $A = A^T$ e $x^T A x > 0 \ \forall x \neq 0$

A è def positiva se $A = A^T$ e $\lambda_i > 0 \ \forall i$



Se $\alpha > 1 \Rightarrow 2\alpha - 2 > 0$

e gli autovalori sono t.c.

$$2\alpha - 2 \leq \lambda \leq 2\alpha + 2$$

$\Rightarrow$ gli autovalori sono positivi $\Rightarrow A$ è def. pos.

Se $\alpha > 1$ $\quad |2\alpha| = 2\alpha > 2$

quindi A è p. diagonale $\Rightarrow$ Jacobi è convergente

$$A = \begin{bmatrix} 2\alpha & 1 \\ 1 & 2\alpha \end{bmatrix}$$

Se $\alpha > 0$

$2\alpha - 2 < 0$ $\quad 0 \in \bigcup R_i$

$$A = \begin{bmatrix} 2\alpha & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2 \end{bmatrix} \qquad \alpha = \frac{1}{2}$$

$$A = \begin{bmatrix} 1 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 1 \end{bmatrix}$$

A è def positiva?

A è def positiva allora A_k è def positiva

$$A_k = A_k^T$$

$$v^T A_k v = \begin{bmatrix} v^T & 0 \end{bmatrix} A \begin{bmatrix} v \\ 0 \end{bmatrix} > 0$$

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ non è definita positiva.

$$A_4 = \begin{bmatrix} 1 & 1 & & \\ 1 & 1 & 1 & \\ & 1 & 1 & 1 \\ & & 1 & 1 \end{bmatrix}$$

Se def pos. Jacobi è applicabile.

$$A = \begin{bmatrix} 2\alpha & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2\alpha \end{bmatrix}$$ Jacobi applicabile $\alpha \neq 0$

A def pos. $\Rightarrow \alpha \neq 0$

A def pos $\Rightarrow A_k$ def pos $\forall k \Rightarrow$

$\Delta_1 = 2\alpha > 0 \qquad \alpha > 0$

A def pos. $\Leftrightarrow$ $A = A^*$, $x^T A x > 0$

$x = e_i \qquad e_i^T A \cdot e_i = a_{ii} = 2\alpha > 0$

$$A = \begin{bmatrix} 2\alpha & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2\alpha \end{bmatrix} \qquad n = 4 \quad \begin{bmatrix} 2\alpha & 1 & & \\ 1 & 2\alpha & 1 & \\ 0 & 1 & 2\alpha & 1 \\ 0 & 0 & 1 & 2\alpha \end{bmatrix}$$

$$x = [1 \; -1, \; 1, \; -1]$$

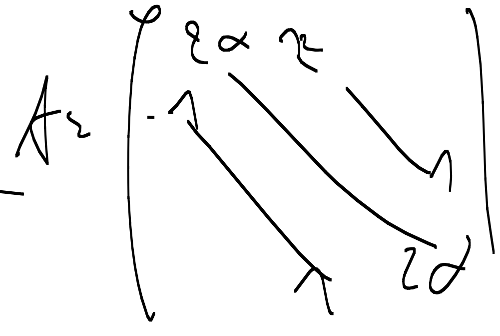
$$x^t A x = [1 \; -1, \; 1 \; -1] \begin{bmatrix} 2\alpha & 1 & & \\ 1 & 2\alpha & 1 & \\ & 1 & 2\alpha & 1 \\ & & 1 & 2\alpha \end{bmatrix}$$

$$= [2\alpha - 1, \; 2 - 2\alpha \quad 2\alpha - 2 \quad 1 - 2\alpha] \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

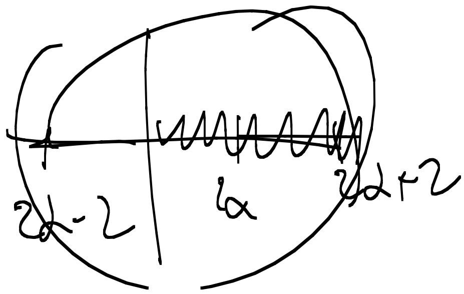
$$(2\alpha - 1) + (2\alpha - 2) + (2\alpha - 2) + (2\alpha - 1) =$$

$$8\alpha - 6 > 0$$

$$8\alpha > 6 \qquad \alpha > \frac{6}{8} = \frac{3}{4} = \frac{n-1}{n}$$

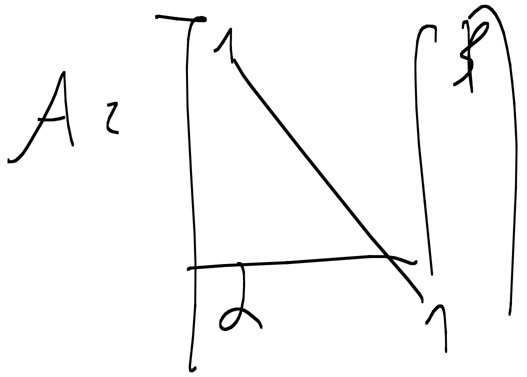


A def pos:



Gli autoval. sono posit. e $\lambda \leq 2\alpha+2$

$\underline{2\alpha - 2 > 0} \Rightarrow 2\alpha > 2 \Rightarrow \alpha > 1$



Per quali α e β la matrice è invertibile?

$|\alpha| < 1$ e $|\beta| < 1$ Ok

perché A è $\beta \cdot \alpha$.

posso determinare $\delta > 1$ tale che A è invertibile $\forall \alpha, \beta$ con $|\alpha| < \delta$ e $|\beta| < \delta$?

No perché se $\alpha = \beta = 1$ A non è invertibile.

$\forall \alpha, \beta \in \mathbb{R}$ A ammette fattorizzazione LU

$$\begin{bmatrix} 1 & & \beta \\ & \ddots & \\ \alpha & & 1 \end{bmatrix} = \left[\begin{array}{c|c} I_{n-1} & \\ \hline \alpha\ 0 \cdots 0 & 1 \end{array}\right] \left[\begin{array}{c|c} I_{n-1} & \begin{matrix}\beta \\ \vdots \\ 0\end{matrix} \\ \hline 0 & 1-\alpha\beta \end{array}\right]$$

$$\alpha\beta + \gamma = 1 \qquad \gamma = 1 - \alpha\beta$$

A è singolare $\iff$ $1 - \alpha\beta = 0 \iff \alpha\beta = 1$

$\alpha \neq 0$ e $\beta = \dfrac{1}{\alpha}$