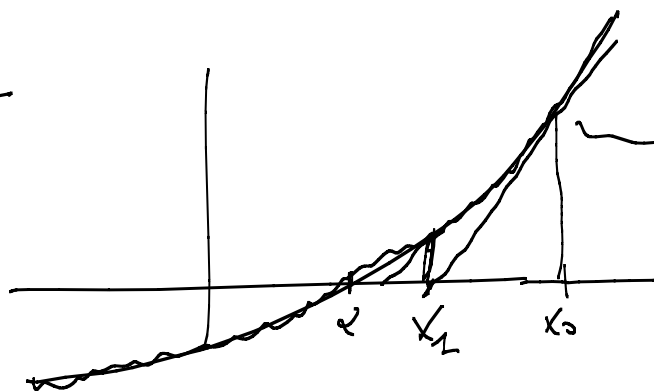


lezione - 18/05

Metodo delle tangenti (Newton)

- il più efficiente metodo di iterazione funzionale.
- semplice di condizioni di convergenza. (espresso in termini di $f(x)$ e non $g(x)$)

Definizione.



retta tangente
al grafico in:
 $(x_0, f(x_0))$

$$\begin{cases} y - f(x_0) = f'(x_0) \cdot (x - x_0) \\ y = 0 \end{cases}$$

$$\Rightarrow x_1 - x_0 = - \frac{f(x_0)}{f'(x_0)}$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Metodo delle
tangenti.

$$\left\{ \begin{array}{l} x_0 \in \mathbb{R} \\ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \end{array} \right.$$

Metodo di Stevezione funzionale $x_{k+1} = g(x_k)$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Convergenza locale.

Teorema: Sia $f \in C^2([a, b])$ $f(\alpha) = 0$
 $f'(\alpha) \neq 0$, $\alpha \in (a, b)$.

Allora il metodo delle tangenti è localmente convergente
in α e la convergenza è almeno quadratica,

Cioè $\exists I_\alpha = [\alpha - \rho, \alpha + \rho]$ tale $\forall x_0 \in I_\alpha$

① $x_k \in I_\alpha$

② $\lim_{k \rightarrow \infty} x_k = \alpha$

| convergenza locale.

$$\textcircled{3} \text{ Se } x_k \neq \alpha \quad \forall k \quad \text{lim} \frac{|x_{k+1} - \alpha|}{|x_k - \alpha|^2} = C \in \mathbb{R}.$$

Dem: $f'(\alpha) \neq 0 \Rightarrow \exists \hat{I}_\alpha \text{ t.c. } f'(x) \neq 0 \quad \forall x \in \hat{I}_\alpha$

$$g: \hat{I}_\alpha \rightarrow \mathbb{R} \quad g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

$$g'(\alpha) = \frac{f(\alpha)f''(\alpha)}{f'(\alpha)^2} = 0 \quad |g'(\alpha)| < 1$$

Condición del teorema del punto fijo. $\Rightarrow \exists \tilde{I}_\alpha \subseteq \hat{I}_\alpha$
 t.c. si lo convergenz local.

Sea $X_0 \in \tilde{I}_\alpha$ e $x_k \neq \alpha \quad \forall k$

$$0 = f(\alpha) = f(x_k) + f'(x_k)(\alpha - x_k) + \frac{f''(\xi_k)(\alpha - x_k)^2}{2}$$

$$x_k - \frac{f(x_k)}{f'(x_k)} = \alpha + \frac{f''(\xi_k)(\alpha - x_k)^2}{2f'(x_k)}$$

$$x_{k+1} = \alpha + \frac{f''(\xi_k)}{f'(x_k)} \frac{(\alpha - x_k)^2}{2}$$

$$\left| \frac{x_{k+1} - \alpha}{(\alpha - x_k)^2} \right| = \left| \frac{1}{2} \frac{f''(\xi_k)}{f'(x_k)} \right|$$

$$\lim_{k \rightarrow +\infty} \frac{|x_{k+1} - \alpha|}{|\alpha - x_k|^2} = \frac{1}{2} \left| \frac{f''(\alpha)}{f'(\alpha)} \right| \in \mathbb{R}.$$

Convergenza in lungo.

Teorema: Sia $f \in C^2([a, b])$ $f(\alpha) = 0$ $\alpha \in (a, b)$.

Esiste $\delta > 0$ tale

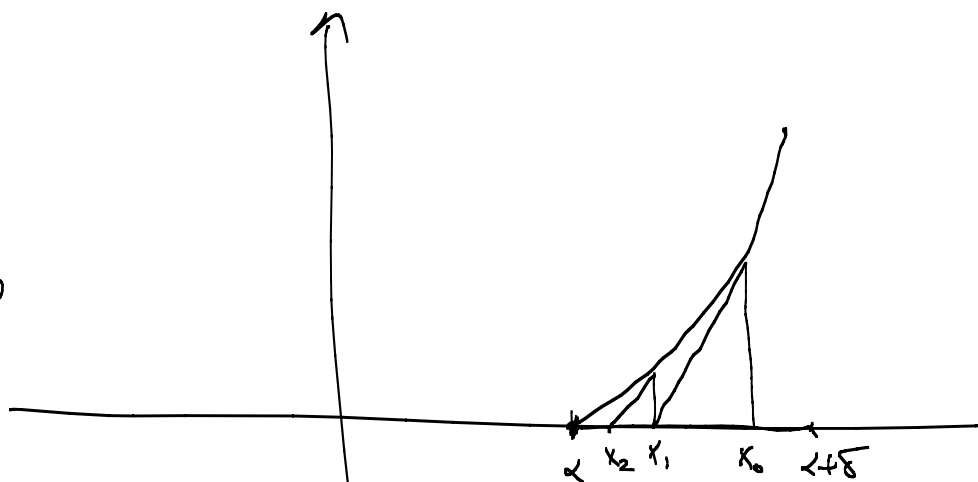
$$\left\{ \begin{array}{l} f'(x) \neq 0 \\ f(x) f''(x) > 0 \end{array} \right. \quad \forall x \in (\alpha, \alpha + \delta] \subseteq [a, b]$$

allora $\forall x_0 \in (\alpha, \alpha + \delta]$ la successione generata dal metodo della tangente a partire da x_0 converge ad α

$$f'(x) > 0$$

$$\Rightarrow f(x) > 0$$

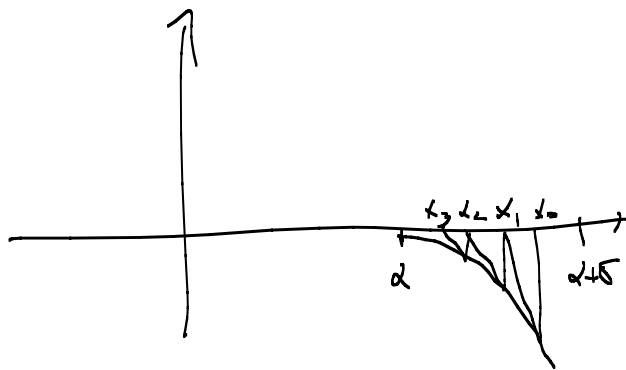
$$\Rightarrow f''(x) > 0$$



$$f'(x) < 0$$

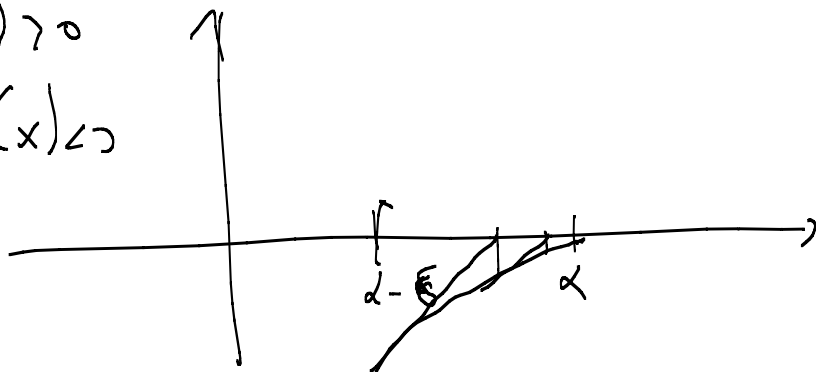
$$\Rightarrow f''(x) < 0$$

$$f''(x) < 0$$



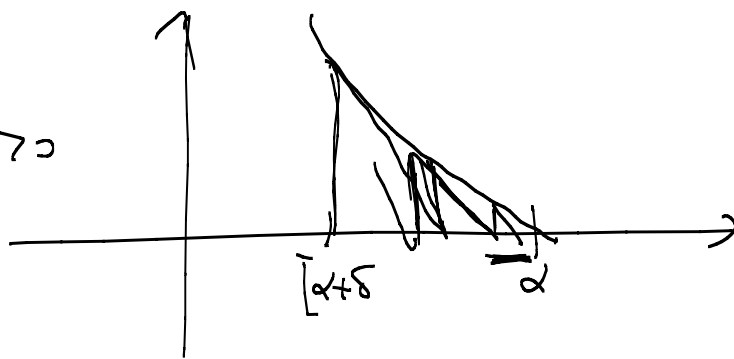
$$f'(x) \neq 0 \quad f'(x) > 0$$

$$f(x) < 0 \Rightarrow f''(x) < 0$$



$$f'(x) < 0$$

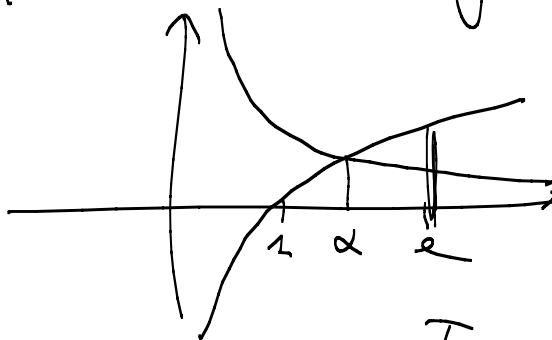
$$f(x) > 0 \quad f''(x) > 0$$



$$f(x) = x \log x - 1 = 0$$

$$\underline{x > 0}$$

$$x \log x - 1 < 0 \Leftrightarrow x \log x < 1 \Leftrightarrow \log x = \frac{1}{x}$$



$$\exists! \alpha \text{ s.t. } f(\alpha) = 0 \quad \underline{\alpha \in (1, e)}$$

$$f(x) = x \log x - 1$$

$$f \in C^\infty(\mathbb{R}^+)$$

$$\lim_{x \rightarrow 0^+} f(x) = ?$$

$$\lim_{x \rightarrow +\infty} f(x) = +\infty$$

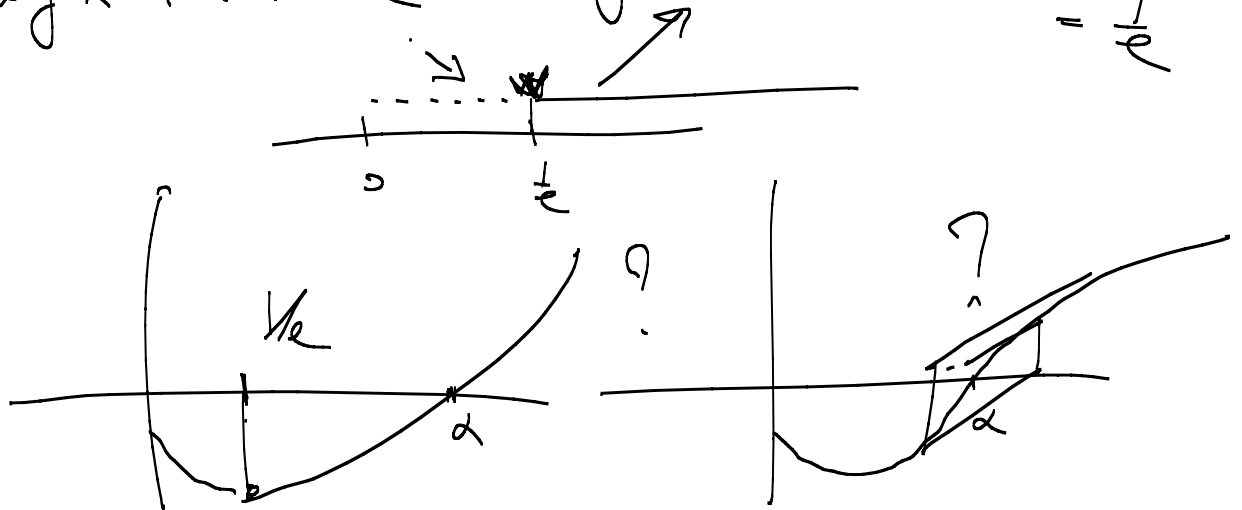
$$\lim_{x \rightarrow 0^+} x \log x = \lim_{x \rightarrow 0^+} \frac{\log x}{\frac{1}{x}} \stackrel{\text{H\u00f6pital}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow 0^+} -\frac{1}{x} \cdot x^2 = 0$$

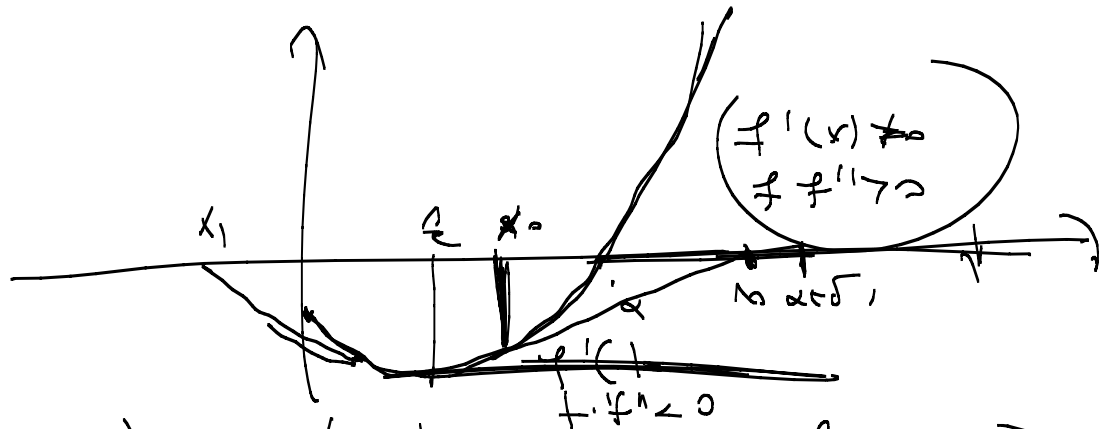
$$\lim_{x \rightarrow 0^+} f(x) = -1$$

$$f'(x) = 1 \cdot \log x + x \cdot \frac{1}{x} = \log x + 1$$

$$\log x + 1 \geq 0 \Leftrightarrow \log x \geq -1 \Leftrightarrow x \geq e^{-1} = \frac{1}{e}$$



$$f''(x) = \frac{1}{x} > 0 \quad \forall x > 0$$



$f \in C^2(\mathbb{R}^+)$ $f'(\alpha) \neq 0 \Rightarrow$ il retto è
 locale convergente in α

$\nabla x_0 \gg \alpha$ il retto della tangente non ancora converge

Per $\frac{1}{\epsilon} < x_0 < \alpha \Rightarrow x_1 \gg \alpha \rightarrow$
 Convergenza.

$$x_1 - \alpha = g(x_0) - g(\alpha) =$$

$$\stackrel{\text{logorismo}}{=} g'(\xi) \cdot (x_0 - \alpha) = \frac{f(\xi) f''(\xi)}{(f'(\xi))^2} (x_0 - \alpha)$$

$$\xi \in [x_0, \alpha]$$

$$x_0 - \alpha < 0$$

$$\frac{f''(\xi) f(\xi)}{(f'(\xi))^2} < 0$$

$$x_1 - \alpha > 0$$

$$\underline{\underline{x_1 > \alpha}}$$

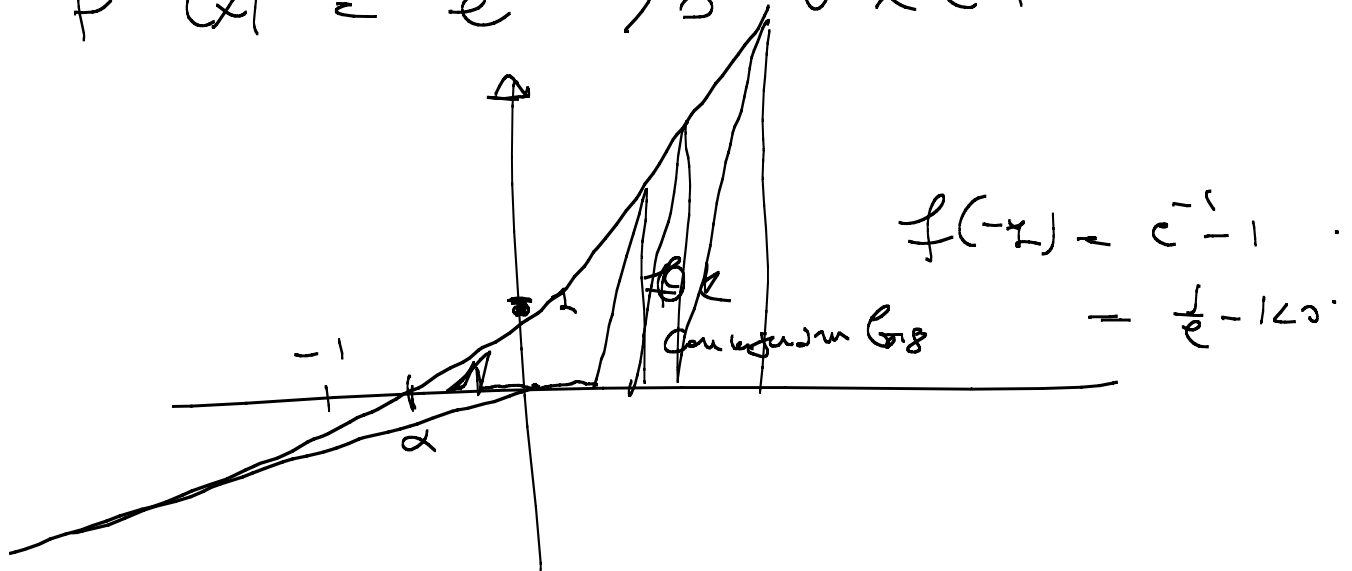
$$f(x) = e^x + x = 0$$

$$f \in C^\infty(\mathbb{R}) \quad \lim_{x \rightarrow +\infty} f(x) = +\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

$$f'(x) = e^x + 1 > 0 \quad \forall x \in \mathbb{R}$$

$$f''(x) = e^x > 0 \quad \forall x \in \mathbb{R}$$



Konvergenzweite h $f'(x) \neq 0$

$$\forall x_0 \in \mathbb{R} \quad x_k \rightarrow \alpha$$

$$f(x) = \frac{1}{x} - a = 0 \quad x = \frac{1}{a} \quad (a \neq 0)$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k + \frac{\left(\frac{1}{x_k} - a\right)}{\frac{1}{x_k^2}}$$

$$x_{k+1} = x_k + x_k^2 \cdot \left(\frac{1}{x_k} - a\right)$$

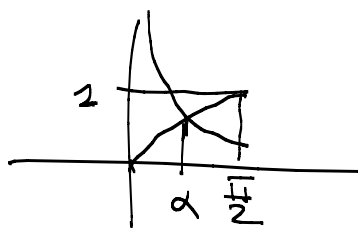
$$= x_k + x_k - a x_k^2$$

$$= 2x_k - a x_k^2 = x_k \cdot (2 - a x_k)$$

b/a $\left(\frac{1}{a}\right)$ cutab dell tangenti

$$b \cdot \frac{1}{a}$$

$$f(x) = \ln x - \frac{1}{x}$$



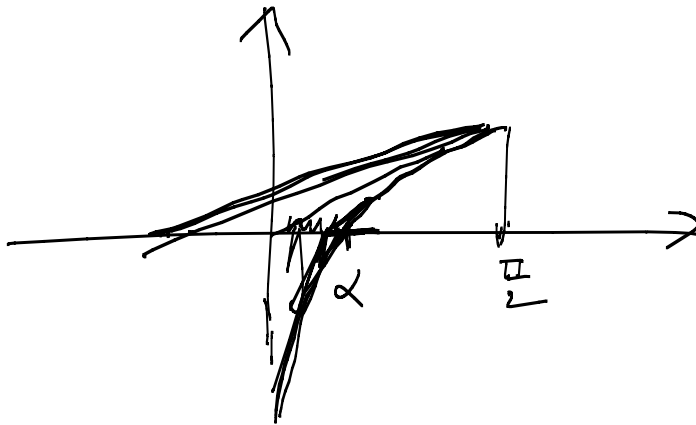
$f_{\text{neu.}} f(x) = -\infty$
 K_{Tot}

$$f\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) - \frac{1}{\frac{\pi}{2}} = 1 - \frac{2}{\pi} > 0$$

$$f \in C^\infty(\mathbb{R}^+)$$

$$f'(x) = \cos x + \frac{1}{x^2} > 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right]$$

$$f''(x) = -\sin x - \frac{2x}{x^3} < -\sin x - \frac{2}{x^2} < 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right]$$



$\frac{\pi}{2}$?

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_0 - \frac{\sin(x_0) - \frac{1}{x_0}}{\cos(x_0) + \frac{1}{x_0^2}} = 0$$

$$x_0 = \frac{f(x_0) - \frac{1}{x_0}}{\cos(x_0) + \frac{1}{x_0^2}}$$

$$x \cos(x) + \frac{1}{x} = f(x) - \frac{1}{x}$$

$$x \cos x - f(x) + \frac{2}{x} = 0$$

$$\sqrt{x^3} - 3x + 1 = 0$$

$$\underline{x \mid 0}$$

$$\sqrt{x^3} - 3x + 1 = 0$$

$$\Leftrightarrow \sqrt{x^3} = 3x - 1$$

$$\Leftrightarrow \frac{x^3 = (3x - 1)^2}{3x - 1 \neq 0 \quad x \neq \frac{1}{3}}$$

$$\sqrt{x} = -2$$

$$\Leftrightarrow \underline{x = 4}$$
