PRODOTTO SCALARE E VETTORIALE

Titolo nota 25/09/2012

Ricevimento Mercoledì 16:30 - 18:30
AULA ?

Codice Esame 193AA

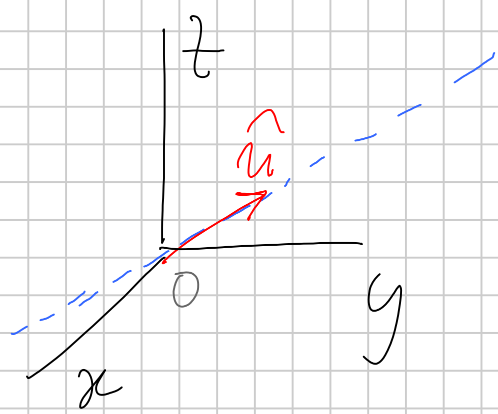
Prodotto <u>per</u> uno scalare

$\underline{v}$ vettore m scalare $\underline{w} = m\underline{v}$

$\hat{u} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$

$\underline{u} = t\hat{u} \quad t \in \mathbb{R}$

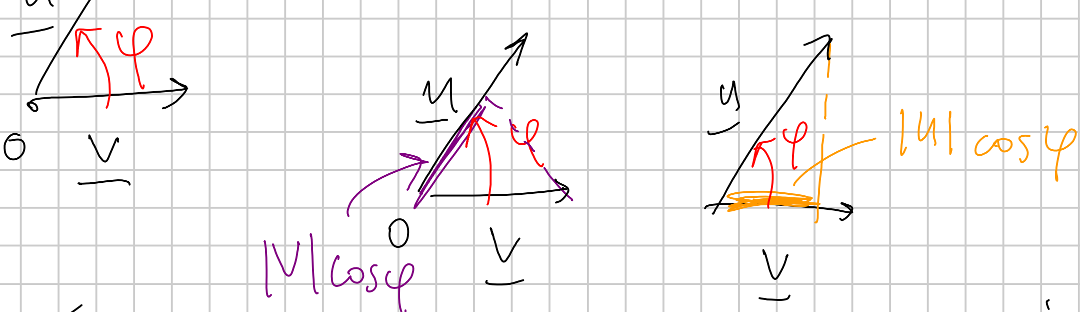
$\alpha^2 + \beta^2 + \gamma^2 = 1$



$$\begin{cases} x = \alpha t \\ y = \beta t \\ z = \gamma t \end{cases}$$

al posto di α, β, γ spesso si trovano l, m, n

$\underline{u} \cdot \underline{v} = |u|\,|v| \cos\varphi$



Se due vettori sono ortogonali $\underline{u}, \underline{v}$, allora $\underline{u} \cdot \underline{v} = 0$

Se $\underline{u} \cdot \underline{v} = 0$ e $|\underline{v}| \neq 0$ $|\underline{u}| \neq 0 \Rightarrow \underline{u} \perp \underline{v}$

Proprietà commutativa $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

$\underline{u} \cdot (\underline{v}_1 + \underline{v}_2) = \underline{u} \cdot \underline{v}_1 + \underline{u} \cdot \underline{v}_2$ Proprietà distributiva

$$\underline{u} \cdot \underline{u} = 0 \iff \underline{u} = \underline{0}$$

$$(m\underline{u}) \cdot \underline{v} = \underline{u} \cdot (m\underline{v}) = m(\underline{u} \cdot \underline{v})$$

$$\underline{v} \cdot \underline{v} = |\underline{v}|^2 = v^2$$

$\hat{\imath}, \hat{\jmath}, \hat{k}$ $\qquad \hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$$

$$\hat{\imath}_h \cdot \hat{\imath}_k = \delta_{hk} = \begin{cases} 1 & \text{se } h = k \\ 0 & \text{se } h \neq k \end{cases}$$

$\hat{\imath}_1 \to \hat{\imath}$

$\hat{\imath}_2 \to \hat{\jmath}$

$\hat{\imath}_3 \to \hat{k}$

Delta di Kronecker

$$\underline{u} = u_x \hat{\imath} + u_y \hat{\jmath} + u_z \hat{k}$$

$$\underline{v} = v_x \hat{\imath} + v_y \hat{\jmath} + v_z \hat{k}$$

$$\underline{u} \cdot \underline{v} = (u_x \hat{\imath} + u_y \hat{\jmath} + u_z \hat{k}) \cdot (v_x \hat{\imath} + v_y \hat{\jmath} + v_z \hat{k}) =$$

$$u_x v_x \underbrace{(\hat{\imath} \cdot \hat{\imath})}_{1} + u_y v_y \underbrace{(\hat{\jmath} \cdot \hat{\jmath})}_{1} + u_z v_z \cdot \underbrace{(\hat{k} \cdot \hat{k})}_{1} + \text{zeri} =$$

$$= u_x v_x + u_y v_y + u_z v_z$$

$$\underline{u} = \sum_{h=1}^{3} u_h \hat{\imath}_h \qquad \underline{v} = v_k \hat{\imath}_k$$

$$\underline{u}\cdot\underline{v} = (u_h \hat{\imath}_h)\cdot(v_k \hat{\imath}_k) = u_h v_k \underbrace{(\hat{\imath}_h \cdot \hat{\imath}_k)}_{\delta_{hk}} =$$

$$u_h v_k \delta_{hk} = \sum_{h=1}^{3} u_h v_h = u_1 v_1 + u_2 v_2 + u_3 v_3$$

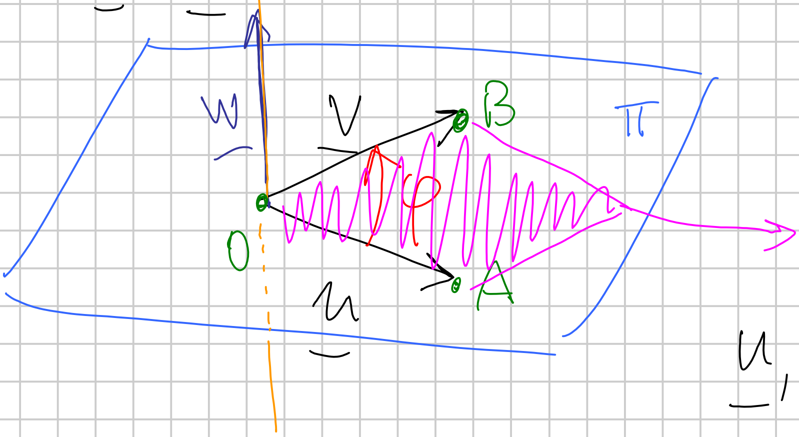
$\hat{v}$ $\qquad \underline{u}\cdot\hat{v} =$ è la componente di $\underline{u}$ lungo $\hat{v}$

$$\underline{u}\cdot\hat{\imath} = u_x \qquad \underline{u}\cdot\hat{\jmath} = u_y \qquad \underline{u}\cdot\hat{k} = u_z$$

PRODOTTO VETTORIALE

2 vettori $\underline{u}$ $\underline{v}$ non paralleli (per ora)

$\underline{u} \wedge \underline{v} = \underline{w}$



$\underline{w}$ è $\perp$ a π

$\underline{w} \perp \underline{u}$, $\underline{w} \perp \underline{v}$

$$\boxed{|\underline{w}| = |\underline{u}|\cdot|\underline{v}|\cdot \operatorname{sen}\varphi}$$

$\underline{u}, \underline{v}, \underline{w}$ devono formare una terna destrorsa

Se uno tra $\underline{v}$ e $\underline{u}$ è $\underline{0}$ oppure se $\underline{u} \parallel \underline{v}$ allora

$\underline{w} = \underline{0}$

Proprietà: $\underline{u} \wedge \underline{v} = -\underline{v} \wedge \underline{u}$ ANTICOMMUTATIVO

$$\underline{u} \wedge (\underline{v}_1 + \underline{v}_2 + \dots \underline{v}_n) = \underline{u} \wedge \underline{v}_1 + \underline{u} \wedge \underline{v}_2 + \dots \underline{u} \wedge \underline{v}_n$$

$$m(\underline{u} \wedge \underline{v}) = (m\underline{u}) \wedge \underline{v} = \underline{u} \wedge (m\underline{v})$$

$\hat{\imath} \wedge \hat{\imath} = \underline{0}$ $\quad \hat{\imath} \wedge \hat{\jmath} = \hat{k}$ terna destrorsa

$\hat{\jmath} \wedge \hat{\jmath} = \underline{0}$ $\quad \hat{\jmath} \wedge \hat{k} = \hat{\imath}$ $\quad \hat{\jmath} \wedge \hat{\imath} = -\hat{k}$

$\hat{k} \wedge \hat{k} = \underline{0}$ $\quad \hat{k} \wedge \hat{\imath} = \hat{\jmath}$ $\quad \hat{k} \wedge \hat{\jmath} = -\hat{\imath}$

$\hat{\imath} \wedge \hat{k} = -\hat{\jmath}$

$$\underline{u} = u_x \hat{\imath} + u_y \hat{\jmath} + u_z \hat{k}$$

$$\underline{v} = v_x \hat{\imath} + v_y \hat{\jmath} + v_z \hat{k}$$

$$\underline{u} \wedge \underline{v} = (u_x \hat{\imath} + u_y \hat{\jmath} + u_z \hat{k}) \wedge (v_x \hat{\imath} + v_y \hat{\jmath} + v_z \hat{k})$$

$$(u_y v_z - u_z v_y)\hat{\imath} + (u_z v_x - u_x v_z)\hat{\jmath} + (u_x v_y - u_y v_x)\hat{k}$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

è un "finto determinante"

$$\boxed{A = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix}$$

$$\boxed{\begin{matrix} 1 \to x \\ 2 \to y \\ 3 \to z \end{matrix}}$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{se } i,j,k \text{ sono una permutazione pari di } 1,2,3 \quad (2,3,1 \text{ o } 3,1,2) \\ -1 & \text{se } i,j,k \text{ sono una perm. dispari di } 1\,3\,2 \quad 2\,1\,3 \quad 3\,2\,1 \\ 0 & \text{se ci sono 2 indici uguali} \end{cases}$$

$A_{ik} = \varepsilon_{ijk}\, u_j$ $\qquad i=1 \quad k=2 \rightarrow j=3$

$132 \rightarrow -u_z$

$i=3, \; k=2$

$$A_{32} = \cancel{\sum_{j=1}^{3}} \varepsilon_{3j2}\, u_j = \varepsilon_{312}\, u_1 = +1 \cdot u_1 = u_x$$

$$(\underline{u} \wedge \underline{v})_i = \cancel{\sum_{j,k}} \varepsilon_{ijk}\, u_j v_k$$

$$\boxed{\varepsilon_{ijk}\, \varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}}$$

Provate a dimostrarlo!

IL PRODOTTO VETTORIALE NON È ASSOCIATIVO

$$\underbrace{(\underline{u} \wedge \underline{v}) \wedge \underline{w}}_{\text{prodotto triplo}} \neq \underline{u} \wedge (\underline{v} \wedge \underline{w})$$

IL PRODOTTO VETTORIALE È MOLTO IMPORTANTE PER CARATTERIZZARE IL PARALLELISMO FRA VETTORI

Se $\underline{u} \parallel \underline{v}$ allora $\underline{u} \wedge \underline{v} = \underline{0}$

Se $\underline{u} \wedge \underline{v} = \underline{0}$ e $\underline{u} \neq \underline{0}$, $\underline{v} \neq \underline{0}$, allora $\underline{u} \parallel \underline{v}$

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \lambda \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \qquad \lambda \neq 0$$

Due modi per imporre il parallelismo.

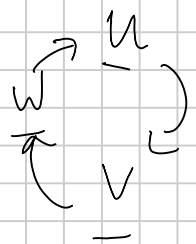
Calcolare $\underline{u} \wedge \underline{v}$ e imporre che sia $= \underline{0}$

$\underline{u} \wedge \underline{x} = \underline{0}$ $\qquad$ $\underline{x} = \lambda \underline{u}$ (per $\lambda = 0$, ho la soluzione ovvia $\underline{x} = \underline{0}$)

$\underline{u} \wedge \underline{x} = \underline{w}$ provate a pensarci!

PRODOTTO MISTO

$$\underline{u} \cdot (\underline{v} \wedge \underline{w}) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$



Prodotto misto $= 0 \iff$ vettori complanari

$$\underline{u} \cdot (\underline{v} \wedge \underline{w}) = \underline{w} \cdot (\underline{u} \wedge \underline{v}) = \underline{v} \cdot (\underline{w} \wedge \underline{u}) =$$

opposti $\qquad$ opposti $\qquad$ opposti

$$= \underline{u} \cdot (\underline{w} \wedge \underline{v}) = \underline{w} \cdot (\underline{v} \wedge \underline{u}) = \underline{v} \cdot (\underline{u} \wedge \underline{w})$$

$$(\underline{v} \wedge \underline{w})_i = \varepsilon_{ijk} v_j w_k =$$

$$= u_\ell (\varepsilon_{ijk} v_j w_k) \delta_{i\ell} = u_\ell \varepsilon_{\ell jk} v_j w_k =$$

$$\varepsilon_{\ell jk} u_\ell v_j w_k$$

$$\underline{a} \cdot \underline{b} = a_h b_k \delta_{hk}$$

DOPPIO PRODOTTO VETTORIALE

$$(\underline{u} \wedge \underline{v}) \wedge \underline{w}$$

$$\underline{u} \wedge \underline{v} = (u_x \hat{i} + u_y \hat{j} + u_z \hat{k}) \wedge (v_x \hat{i} + v_y \hat{j} + v_z \hat{k})$$

$$[(u_y v_z - u_z v_y)\hat{i} + (u_z v_x - u_x v_z)\hat{j} + (u_x v_y - u_y v_x)\hat{k}] \wedge (w_x \hat{i} + w_y \hat{j} + w_z \hat{k}) =$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_y v_z - u_z v_y & u_z v_x - u_x v_z & u_x v_y - u_y v_x \\ w_x & w_y & w_z \end{vmatrix} =$$

$$= (u_z v_x w_z - u_x v_z w_z - u_x v_y w_y + u_y v_x w_y)\hat{i}$$
$$+ (u_x v_y w_x - u_y v_x w_x - u_y v_z w_z + u_z v_y w_z)\hat{j}$$
$$+ (u_y v_z w_y - u_z v_y w_y - u_z w_x v_x + u_x w_x v_z)\hat{k}$$

$$\boxed{\begin{aligned}&-v_x w_x (u_y \hat{j} + u_z \hat{k} + u_x \hat{\imath})\\ &-v_y w_y (u_x \hat{\imath} + u_z \hat{k} + u_y \hat{j})\\ &-v_z w_z (u_x \hat{\imath} + u_y \hat{j} + u_z \hat{k})\end{aligned}} \quad \boxed{\begin{aligned}&+ v_x w_x u_x \hat{\imath}\\ &+ v_y w_y u_y \hat{j} + \text{altri 6 pezzi}\\ &+ v_z w_z u_z \hat{k}\end{aligned}}$$

$$-(\underline{v} \cdot \underline{w})\, \underline{u} \qquad\qquad + (\underline{w} \cdot \underline{u})\, \underline{v}$$

$$[(\underbrace{u \wedge v}_{T_i}) \wedge w]_\ell = \qquad \boxed{\varepsilon_{\ell i n} = \varepsilon_{i n \ell}}$$

$$\varepsilon_{\ell i n} \underbrace{(\varepsilon_{ijk} u_j v_k)}_{(u \wedge v)_i} w_n = \varepsilon_{\ell i n} \varepsilon_{ijk} u_j v_k w_n =$$

$$= \underbrace{\varepsilon_{i n \ell} \varepsilon_{ijk}}_{\text{posso sommare su } i} u_j v_k w_n =$$

$$= (\delta_{nj} \delta_{\ell k} - \delta_{nk} \delta_{\ell j}) u_j v_k w_n =$$

$$= u_n v_\ell w_n - u_\ell v_n w_n = \overset{u \cdot v}{(u_n w_n)} v_\ell - \overset{v \cdot w}{(v_n w_n)} u_\ell$$

$$= (\underline{u} \cdot \underline{w}) v_\ell - (\underline{v} \cdot \underline{w}) u_\ell \qquad \forall \ell = 1, 2, 3$$

$$(\underline{u} \wedge \underline{v}) \wedge \underline{w} = (\underline{u} \cdot \underline{w}) \underline{v} - (\underline{v} \cdot \underline{w}) \underline{u} \qquad \text{OK}$$

VETTORI DIPENDENTI DA UN PARAMETRO

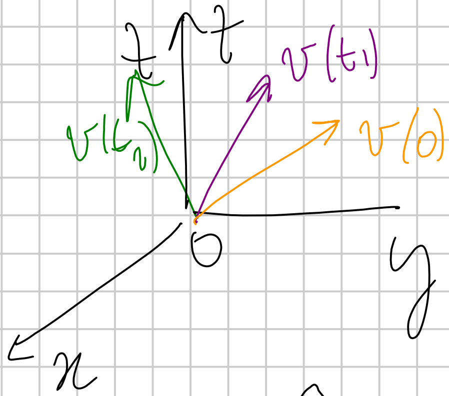
→ CURVE NEL PIANO E NELLO SPAZIO

$I = [t_1, t_2]$ $\quad f : I \to \mathbb{R}$

$\underline{v}(t)$

$\forall\ t \in I$ associamo un vettore $\underline{v}(t)$

$\hat{u}_t = \underline{u}(t)$



$\underline{v}(t) = v_x(t)\,\hat{i} + v_y(t)\,\hat{j} + v_z(t)\,\hat{k}$

Se le 3 funzioni $v_x : I \to \mathbb{R}$ $\quad v_y : I \to \mathbb{R}$ $\quad v_z : I \to \mathbb{R}$ sono continue in I, si dirà che $\underline{v}(t)$ è continuo in I

$$\lim_{t \to t_0} \underline{v}(t) = \underline{v}(t_0) \qquad \text{continuità}$$

$$\lim_{t \to t_0} \underline{v}(t) = \underline{\bar{v}}$$ $\exists$ il limite, ma non coincide con $\underline{v}(t_0)$

Derivata di un vettore

$$\lim_{\Delta t \to 0} \frac{\underline{v}(t + \Delta t) - \underline{v}(t)}{\Delta t} = \underline{v}'(t)$$

$\hat{r}$ direzione $\quad v_r = \underline{v} \cdot \hat{r} \quad \hat{r}$ fisso

$v_r(t) = \underline{v}(t) \cdot \hat{r}$ scalare

$$\frac{dv_r}{dt} = \left(\frac{d\underline{v}}{dt}\right) \cdot \hat{u}$$

$$\frac{dv_r}{dt} = \lim_{\Delta t \to 0} \frac{v_r(t+\Delta t) - v_r(t)}{\Delta t} =$$

$$= \lim_{\Delta t \to 0} \frac{\underline{v}(t+\Delta t) \cdot \hat{u} - \underline{v}(t) \cdot \hat{u}}{\Delta t} =$$

$$= \lim_{\Delta t \to 0} \frac{[\underline{v}(t+\Delta t) - \underline{v}(t)] \cdot \hat{u}}{\Delta t}$$

ESCE DAL LIMITE

$$= \left(\lim_{\Delta t \to 0} \frac{\underline{v}(t+\Delta t) - \underline{v}(t)}{\Delta t}\right) \cdot \hat{u} = \left(\frac{d\underline{v}}{dt}\right) \cdot \hat{u}$$

$$= \left(\frac{d\underline{v}}{dt}\right)_u$$

$$\frac{d\underline{v}}{dt} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} + \frac{dv_z}{dt}\hat{k}$$

$$\frac{d}{dt}\sum_{i=1}^{n} \underline{v}_i(t) = \sum_{i=1}^{n} \frac{d\underline{v}_i}{dt}$$

$$\frac{d}{dt}\left[f(t)\,\underline{v}(t)\right] = \frac{df}{dt}\,\underline{v}(t) + f(t)\cdot\frac{d\underline{v}(t)}{dt}$$

$$\frac{d}{dt}(\underline{v}\cdot\underline{u}) = \frac{d\underline{v}}{dt}\cdot\underline{u} + \underline{v}\cdot\frac{d\underline{u}}{dt}$$

$$\frac{d}{dt}(\underline{v}\wedge\underline{u}) = \frac{d\underline{v}}{dt}\wedge\underline{u} + \underline{v}\wedge\frac{d\underline{u}}{dt}$$

Se $|\underline{v}|$ è costante, allora $\frac{d\underline{v}}{dt} \perp \underline{v}$
(vero in particolare per i versori)

$$\underline{v}\cdot\underline{v} = \text{costante}$$

$$0 = \frac{d}{dt}(\underline{v}\cdot\underline{v}) = \underline{v}\cdot\frac{d\underline{v}}{dt} + \frac{d\underline{v}}{dt}\cdot\underline{v} = 2\,\underline{v}\cdot\frac{d\underline{v}}{dt}$$

$$\underline{v}\cdot\frac{d\underline{v}}{dt} = 0 \qquad \underline{v}\perp\frac{d\underline{v}}{dt}$$